

DC Josephson Effect in a Tomonaga-Luttinger Liquid

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The dc Josephson effect in a one-dimensional Tomonaga-Luttinger (TL) liquid is studied on the basis of two bosonized models. We first consider a TL liquid sandwiched between two superconductors with a strong barrier at each interface. Both the interfaces are assumed to be perfect if the barrier potential is absent. We next consider a TL liquid with open boundaries, weakly coupled with two superconductors. Without putting strong barriers, we instead assume that the coupling at each interface is described by a tunnel junction. We calculate the Josephson current in each model, and find that the two models yield same results. The Josephson current is suppressed by repulsive electron-electron interactions. It is shown that the suppression is characterized by only the correlation exponent for the charge degrees of freedom. This result is inconsistent with a previously reported result, where the spin degrees of freedom also affects the suppression. The reason of this inconsistency is discussed.

1. Introduction

Progress in micro-fabrication technology has enabled us to prepare normal-conductor-superconductor (NS) composite systems without a Schottky barrier at the NS interface.¹ A highly controllable normal segment has been realized by using a two-dimensional electron system formed at semiconductor heterostructures.² We are interested in how the electronic transport properties are influenced by superconducting proximity effect. Particular attention has been focused on the low-temperature transport properties with the phase coherence of electrons in the normal segment. Various experiments have been performed to reveal unusual features of the conductance in NS systems and the Josephson effect in SNS systems.^{2,3}

Progress in micro-fabrication technology has also made it possible to prepare a very narrow quantum wire which can be regarded as an idealistic one-dimensional (1D) electron system.^{4,5} The electron transport in a 1D system with a few barriers has been studied extensively.⁶⁻¹⁴ The central problem addressed there is how mutual electron-electron interactions affect the transport properties. Electron-electron interactions greatly affect the low-energy properties of a 1D electron system. A 1D interacting electron system with a gapless excitation is generally described as a Tomonaga-Luttinger (TL) liquid.¹⁵⁻¹⁷ The strength of electron-electron interactions is characterized by the correlation exponents K_ρ and K_σ , where K_ρ (K_σ) is related to the charge (spin) degrees of freedom. For example, $K_\rho < 1$ and $K_\sigma = 1$ corresponds to the spin-independent repulsive interaction cases, and the system is reduced to the noninteracting electron gas when $K_\rho = K_\sigma = 1$. Reflecting the nature of the TL liquid, we expect that the electron transport in a 1D system shows an anomalous property, which is not seen in a conventional Fermi liquid. Kane and Fisher studied the electron transport in a spin-less TL liquid with a single barrier potential of δ -function type.⁶ Using renormalization-group argument, they found that for re-

pulsive electron-electron interactions with $K_\rho < 1$, the potential becomes large after renormalization even if it initially is very small. This means that electrons at low energy see the potential as if it were effectively infinite. Thus, the low-energy properties are described by the open-boundary fixed point, at which the 1D system is effectively disconnected at the barrier. This anomalous behavior strongly affects the transport through the barrier. It is shown that the conductance G is suppressed with decrease of temperature T as $G \propto T^{2K_\rho^{-1}-2}$, and eventually vanishes at $T = 0$. This is a clear manifestation of the TL-liquid behavior. The extension to spin-dependent cases is achieved by Kane and Fisher,⁸ and Furusaki and Nagaosa.¹⁰ The conductance in this case also obeys a power-law behavior as a function of T , and its exponent is determined by K_ρ and K_σ . Furthermore, the electron transport in a TL liquid with a double-barrier structure was also studied by these authors.⁷⁻⁹

Inspired by the developments mentioned above, considerable attention has been attracted to the transport properties of a TL liquid coupled with superconductors.¹⁸⁻²⁷ The central problem is how the superconducting proximity effect in a 1D system is modified by electron-electron interactions. Particular interest is focused on the Josephson effect in a superconductor-TL-liquid-superconductor (STLLS) system,^{19-21,24,26,27} which is the issue of the present paper.

Let us consider a clean 1D noninteracting electron system of length L sandwiched between superconductors (see Fig. 1(a)). We focus our attention on the long-junction case, where v_F/L (v_F : Fermi velocity) is much smaller than the energy gap Δ of superconductors. In the low-temperature regime of $T \ll v_F/L$, the Josephson critical current obeys $j_c \propto ev_F/L$.²⁸ In the case of an STLLS system, we expect that the critical current deviates from the inversely linear L -dependence due to the TL-liquid behavior. This problem is first discussed by Fazio *et al.*^{19,20} They considered a clean TL liquid of infinite length weakly connected to two superconductors

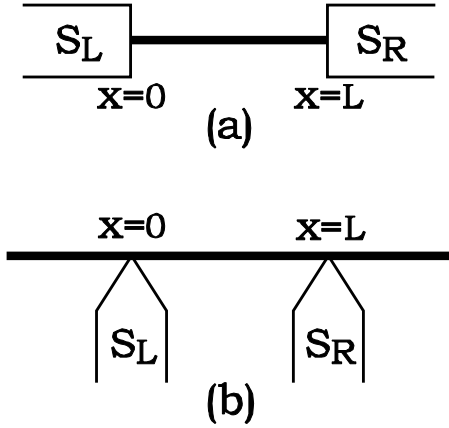


Fig. 1. The geometries discussed in the text. (a) One-dimensional electron system of length L sandwiched between two superconductors. (b) One-dimensional electron system of infinite length weakly coupled with two superconductors. S_L (S_R) represents the left (right) superconductor.

(see Fig. 1(b)), where the separation between the two NS contacts is L . They found that when $T \ll v_F/L$, the L -dependence of the critical current is given by $j_c^{\text{FHO}} \propto (1/L)^{K_\rho^{-1} + K_\sigma - 1}$. This power-law L -dependence is a clear manifestation of the TL-liquid behavior. It indicates that as long as $K_\rho^{-1} + K_\sigma > 2$, the critical current is suppressed by electron-electron interactions compared with the noninteracting case. This result shows that both the charge and spin degrees of freedoms influence the suppression of the critical current. It should be noted that in calculating the critical current in the STLLS system, Fazio *et al.* completely neglected potential scattering at the contacts between the TL liquid and the superconductors. Fabrizio and Gogolin²⁹ pointed out that since the potential induced by the contacts becomes large after renormalization even if it initially is very small,⁶ the TL liquid is finally disconnected at the NS contacts. They suggested that in such a situation, we should employ a TL liquid with open boundaries to correctly take account of the potential scattering in studying the Josephson effect. Maslov *et al.*²¹ studied a different model, in which a TL liquid of length L is sandwiched between superconductors (see Fig. 1(a)). They presented a useful bosonized description of the STLLS system, and calculated the Josephson current when the NS interfaces are perfect. They also examined the L -dependence of the critical current in the presence of a strong barrier at each NS interface. However, they do not correctly take account of NS boundary condition in their renormalization-group argument. The works mentioned above treat a TL liquid with repulsive electron-electron interactions. The Josephson effect in attractive interaction cases has been studied extensively by Affleck *et al.*²⁶

In this paper, we study the dc Josephson effect in a TL liquid using two different models. We first consider a TL liquid of length L sandwiched between two superconductors with a barrier at each NS interface. Both the NS interfaces are assumed to be perfect if the barrier potential is absent. We call it the interface-barrier model.

Particular attention is focused on the limit where the barrier potential is very strong. This model is equivalent to that treated by Maslov *et al.*²¹ We next consider a TL liquid of length L with open boundaries, which is weakly connected at its left (right) end with the left (right) superconductor through a tunnel junction. We call it the weak-coupling model. This model is equivalent to that suggested by Fabrizio and Gogolin.²⁹ It is shown that the two models provide us essentially the same expression of the Josephson current in both the high-temperature regime of $v_F/L \ll T \ll \Delta$ and the low-temperature regime of $T \ll v_F/L$. We find that $j_c \propto (1/L)^{2K_\rho^{-1}-1}$ in the low-temperature regime. Clearly, this result is inconsistent with Fazio *et al.*'s result,^{19,20} where the L -dependence is characterized by both K_ρ and K_σ . The inconsistency is attributed to the difference in the treatment of the potential induced by the NS contacts. The basic assumption of Fazio *et al.* is that the superconductors do not influence the uniformity of the potential along the 1D system, while we assume that the low-energy property of an STLLS system is described by the open-boundary fixed point. Since the potential induced by the contacts becomes large after renormalization even if it initially is very small, Fazio *et al.*'s result is unstable against the renormalization unless the induced potential is completely negligible. If the induced potential cannot be neglected, the 1D system flows to the open-boundary fixed point. That is, the 1D system is effectively disconnected at the NS contacts. Fazio *et al.*'s model is not appropriate in this situation, and we expect that our weak-coupling model provides us correct results.

This paper is organized as follows. In the next section, we present the interface-barrier model on the basis of a bosonized description of an STLLS system. The bosonization procedure based on Haldane's argument is outlined in Appendix A. In §3, we calculate the Josephson current in the interface-barrier model using the instanton approximation. The Josephson current in the weak-coupling model is studied in §4. We employ the bosonized description of a TL liquid with open boundaries given by Fabrizio and Gogolin,²⁹ and calculate the Josephson current within the lowest-order perturbation with respect to the coupling between the TL liquid and the superconductors. Section 5 is devoted to discussion. The argument concerning to the interface-barrier model has been briefly reported in ref. 24. Some errors in ref. 24 are corrected in this paper. We set $\hbar = k_B = 1$ throughout the paper.

2. Interface-Barrier Model

We consider a 1D electron system of length L sandwiched between superconductors (see Fig. 1(a)). The superconducting order is characterized by the pair potential $\Delta(x)$. We assume that $\Delta(x)$ is given by²⁸

$$\Delta(x) = \begin{cases} \Delta e^{i\chi_1} & \text{for } x \leq 0 \\ 0 & \text{for } 0 < x < L \\ \Delta e^{i\chi_2} & \text{for } L \leq x, \end{cases} \quad (1)$$

where χ_1 (χ_2) is the macroscopic phase of the left (right) superconductor. We assume that the length L is much longer than the coherence length which is defined by

$\xi = v_F/\Delta$ with the Fermi velocity v_F . A quasiparticle state in our system is described by the Bogoliubov-de Gennes (BdG) equation.³⁰ Due to Andreev reflection at NS interfaces, quasiparticle states depend on the phase difference $\chi = \chi_2 - \chi_1$. We introduce a set of eigenfunctions of the BdG equation. The eigenfunctions are easily obtained if we assume that perfect Andreev reflection (i.e., zero normal scattering) is achieved at the NS interfaces. In the perfect Andreev reflection case, we obtain the low-energy eigenfunctions in the Nambu notation

$$\begin{pmatrix} u_q(x) \\ v_q(x) \end{pmatrix} = \frac{1}{\sqrt{2L}} \begin{pmatrix} e^{-i\eta_+} e^{iq_+(x+\xi/2)} \\ e^{i\eta_+} e^{-iq_+(x+\xi/2)} \end{pmatrix} e^{ik_F x}, \quad (2)$$

$$\begin{pmatrix} f_q(x) \\ g_q(x) \end{pmatrix} = \frac{1}{\sqrt{2L}} \begin{pmatrix} e^{i\eta_-} e^{-iq_-(x+\xi/2)} \\ e^{-i\eta_-} e^{iq_-(x+\xi/2)} \end{pmatrix} e^{-ik_F x}, \quad (3)$$

where $q = \pi(n + 1/2)/(L + \xi)$ with $n = 0, \pm 1, \pm 2, \dots$ and

$$q_{\pm} = q \pm \frac{\chi}{2(L + \xi)}, \quad (4)$$

$$\eta_{\pm} = \frac{\pi}{4} \mp \frac{\chi_1}{2}. \quad (5)$$

We have assumed that the Fermi wave number k_F is given by $k_F = \pi n_0/(L + \xi)$, where n_0 is an integer. Let ϵ_{q+} and ϵ_{q-} be the linearized eigenenergy of the state eq. (2) and that of the state eq. (3), respectively. They are given by

$$\epsilon_{q\pm} = v_F q_{\pm} + \frac{1}{2m} \left(\frac{\chi}{2(L + \xi)} \right)^2. \quad (6)$$

In deriving eqs. (2) and (3), we have used the following approximation

$$\frac{\epsilon_{q\pm} + i\sqrt{\Delta^2 - \epsilon_{q\pm}^2}}{\Delta} = \exp i \left(\frac{\pi}{2} - \frac{\epsilon_{q\pm}}{\Delta} \right), \quad (7)$$

which is justified when $\epsilon_{q\pm} \ll \Delta$.

We assume that any quasiparticle state in our system is described by eqs. (2) or (3). This is the key approximation allowing us to perform the bosonization procedure. Clearly, this cannot be justified for q of the order of, or greater than, ξ^{-1} . However, high-energy states compared with Δ play only a minor role in low-energy properties, which are of interest to us. To study Josephson effect, we must take account of the zero modes¹⁷ as well as the nonzero modes. To do so, we introduce the winding-number operators J and M , and the zero-mode operators φ_ρ and φ_σ , which satisfy $[J, \varphi_\rho] = [M, \varphi_\sigma] = 2i$ and $[J, \varphi_\sigma] = [M, \varphi_\rho] = 0$. The operator J (M) is related to the excess charge (spin) accumulated in the 1D system. The nonzero modes are described by the boson operators α_q , α_q^\dagger , β_q and β_q^\dagger . The bosonization procedure is outlined in Appendix A.

In terms of the operators introduced above, the bosonized Hamiltonian is given by

$$H = \frac{\pi}{4L} \left(v_\rho K_\rho \left(J + \frac{\chi}{\pi} \right)^2 + \frac{v_\sigma}{K_\sigma} M^2 \right) + \sum_{q>0} (v_\rho q \alpha_q^\dagger \alpha_q + v_\sigma q \beta_q^\dagger \beta_q), \quad (8)$$

where v_ρ (v_σ) is the velocity of charge (spin) excitation and $q = \pi(n + 1/2)/(L + \xi)$ with $n = 0, 1, 2, \dots$. In eq. (8),

K_ρ and K_σ are the correlation exponents, which characterize the interaction strengths. For example, $K_\rho < 1$ and $K_\sigma = 1$ corresponds to the spin-independent repulsive interaction case, and the system is reduced to the noninteracting electron gas when $K_\rho = K_\sigma = 1$. It is important to note that eigenvalues of $J + M$ are limited to be even integers, as shown in Appendix A.

We express the electron-field operator as $\psi_s(x)$, where $s = \pm$ denotes spin index, and decompose it as

$$\psi_s(x) = \psi_{1s}(x) + \psi_{2s}(x), \quad (9)$$

where ψ_{1s} and ψ_{2s} represent the right and left movers, respectively. To express ψ_{1s} and ψ_{2s} , we define the phase fields:³¹

$$\Theta_+(x) = \varphi_\rho + \theta_+(x), \quad (10)$$

$$\Theta_-(x) = \frac{\pi}{L} \left(J + \frac{\chi}{\pi} \right) \left(x + \frac{\xi}{2} \right) + \theta_-(x), \quad (11)$$

$$\Phi_+(x) = \frac{\pi}{L} M \left(x + L + \frac{\xi}{2} \right) + \phi_+(x), \quad (12)$$

$$\Phi_-(x) = \varphi_\sigma + \phi_-(x). \quad (13)$$

The nonzero-mode components θ_{\pm} and ϕ_{\pm} , which describe low-energy fluctuations around the zero modes, are expressed in terms of the boson operators as

$$\theta_+(x) = i\sqrt{K_\rho} \sum_{q>0} \sqrt{\frac{2\pi}{qL}} e^{-\alpha q/2} \cos q \left(x + \frac{\xi}{2} \right) (a_q^\dagger - a_q), \quad (14)$$

$$\theta_-(x) = \frac{1}{\sqrt{K_\rho}} \sum_{q>0} \sqrt{\frac{2\pi}{qL}} e^{-\alpha q/2} \sin q \left(x + \frac{\xi}{2} \right) (a_q^\dagger + a_q), \quad (15)$$

$$\phi_+(x) = \sqrt{K_\sigma} \sum_{q>0} \sqrt{\frac{2\pi}{qL}} e^{-\alpha q/2} \sin q \left(x + \frac{\xi}{2} \right) (b_q^\dagger + b_q), \quad (16)$$

$$\phi_-(x) = \frac{i}{\sqrt{K_\rho}} \sum_{q>0} \sqrt{\frac{2\pi}{qL}} e^{-\alpha q/2} \cos q \left(x + \frac{\xi}{2} \right) (b_q^\dagger - b_q), \quad (17)$$

where α is a positive infinitesimal. The phase field Θ_+ is regarded as the phase of the charge density wave, while Θ_- corresponds to the Josephson phase. The phase fields Φ_+ and Φ_- are related to the spin degrees of freedom. Using the phase fields, we can express ψ_{1s} and ψ_{2s} as³²

$$\psi_{1s}(x) = \frac{1}{\sqrt{2\pi\alpha}} \exp \left(ik_F x + \frac{i}{2} (\Theta_+(x) + \Theta_-(x) + s\Phi_+(x) + s\Phi_-(x)) \right), \quad (18)$$

$$\psi_{2s}(x) = \frac{1}{\sqrt{2\pi\alpha}} \exp \left(-ik_F x + \frac{i}{2} (-\Theta_+(x) + \Theta_-(x) - s\Phi_+(x) + s\Phi_-(x)) \right). \quad (19)$$

Now we put a barrier at each NS interface. For simplicity, we assume that both the barriers have the same

strength. Then, the corresponding Hamiltonian is

$$H^B = -V_0 \sum_s \left(\left(\psi_{1s}^\dagger(0) \psi_{2s}(0) + \text{h.c.} \right) + \left(\psi_{1s}^\dagger(L) \psi_{2s}(L) + \text{h.c.} \right) \right). \quad (20)$$

To express H^B , it is convenient to introduce the new phase variables:^{8,9}

$$\bar{\theta} = \frac{1}{2} (\theta_+(L) + \theta_+(0)), \quad (21)$$

$$\tilde{\theta} = (\theta_+(L) - \theta_+(0)), \quad (22)$$

$$\bar{\phi} = \frac{1}{2} (\phi_+(L) + \phi_+(0)), \quad (23)$$

$$\tilde{\phi} = (\phi_+(L) - \phi_+(0)). \quad (24)$$

The time-derivative of $\bar{\theta}$ ($\bar{\phi}$) is related to the charge (spin) current, while $\tilde{\theta}$ ($\tilde{\phi}$) is related to the excess charge (spin) accumulated in the 1D system. In terms of them, we find that

$$H^B = -\frac{4V_0}{\pi\alpha} \left[\cos(\varphi_\rho + \bar{\theta} + k_F L) \cos\left(\frac{\tilde{\theta}}{2} + k_F L\right) \times \cos\left(\bar{\phi} + \frac{3}{2}\pi M\right) \cos\left(\frac{\tilde{\phi}}{2} + \pi M\right) + \sin(\varphi_\rho + \bar{\theta} + k_F L) \sin\left(\frac{\tilde{\theta}}{2} + k_F L\right) \times \sin\left(\bar{\phi} + \frac{3}{2}\pi M\right) \sin\left(\frac{\tilde{\phi}}{2} + \pi M\right) \right], \quad (25)$$

where we approximated as $(3\pi M/2)(1 + \xi/(3L)) \approx 3\pi M/2$.

3. Josephson Current in the Interface-Barrier Model

In this section, we consider the interface-barrier model in the limit where the barrier potential is very strong. The total Hamiltonian of our system is $H_{\text{total}} = H + H^B$. We calculate the partition function Z in terms of an imaginary-time path integral. The dc Josephson current is obtained through the well-known relation

$$j(\chi) = -2eT \frac{\partial \ln Z}{\partial \chi}. \quad (26)$$

Since M commutes with H^B , then M remains a conserved quantity despite the presence of the barriers at the NS interfaces. By contrast, J is not a good quantum number when $V_0 \neq 0$. The partition function is expressed as

$$Z = \sum_{\substack{M=-\infty \\ (M=\text{even})}}^{\infty} \exp\left(-\beta \frac{\pi v_\sigma M^2}{4K_\sigma L}\right) Z_+ + \sum_{\substack{M=-\infty \\ (M=\text{odd})}}^{\infty} \exp\left(-\beta \frac{\pi v_\sigma M^2}{4K_\sigma L}\right) Z_-, \quad (27)$$

where

$$Z_\pm = \sum_{m=-\infty}^{\infty} \int_{\Delta\varphi_\rho=2\pi m} \mathcal{D}\varphi_\rho \int \prod_{q>0} \mathcal{D}a_q^\dagger \mathcal{D}a_q \mathcal{D}b_q^\dagger \mathcal{D}b_q \times \exp\left(i \int_0^\beta d\tau \frac{\chi + \zeta_\pm}{2\pi} \partial_\tau \varphi_\rho - S^Z - S^{\text{NZ}} - S^B\right), \quad (28)$$

with $\zeta_+ = 0$ and $\zeta_- = \pi$, and

$$S^Z = \int_0^\beta d\tau \frac{L}{4\pi v_\rho K_\rho} (\partial_\tau \varphi_\rho)^2, \quad (29)$$

$$S^{\text{NZ}} = \int_0^\beta d\tau \sum_{q>0} (a_q^\dagger \partial_\tau a_q + v_\rho q a_q^\dagger a_q + b_q^\dagger \partial_\tau b_q + v_\sigma q b_q^\dagger b_q), \quad (30)$$

$$S^B = \int_0^\beta d\tau H^B(\tau). \quad (31)$$

In eq. (28), the integration over φ_ρ is carried out under the boundary condition of $\varphi_\rho(\beta) - \varphi_\rho(0) = 2\pi m$. This boundary condition is imposed to take the discrete nature of J into account. Furthermore, we used ζ_\pm to maintain the constraint $J + M = \text{even}$.³³

Before considering the strong-barrier limit, which is of interest to us, we treat the case of $V_0 \rightarrow 0$ to examine the validity of eq. (27). When $S^B = 0$, the partition function factorizes as $Z = Z(\chi) \bar{Z}$, where $Z(\chi)$ and \bar{Z} are the contribution from the zero modes and that from the nonzero modes, respectively. The former contribution \bar{Z} is given by

$$\bar{Z} = \sum_{\substack{M=-\infty \\ (M=\text{even})}}^{\infty} \exp\left(-\beta \frac{\pi v_\sigma M^2}{4K_\sigma L}\right) \bar{Z}_+ + \sum_{\substack{M=-\infty \\ (M=\text{odd})}}^{\infty} \exp\left(-\beta \frac{\pi v_\sigma M^2}{4K_\sigma L}\right) \bar{Z}_-, \quad (32)$$

where

$$\bar{Z}_\pm = \sum_{m=-\infty}^{\infty} \int_{\Delta\varphi_\rho=2\pi m} \mathcal{D}\varphi_\rho \times \exp\left(i \int_0^\beta d\tau \frac{\chi + \zeta_\pm}{2\pi} \partial_\tau \varphi_\rho - S^Z\right). \quad (33)$$

The latter contribution \tilde{Z} , which does not depend on χ , is irrelevant for the Josephson effect. We decompose φ_ρ as $\varphi_\rho(\tau) = \varphi_{\rho 0} + 2\pi m\tau/\beta + \tilde{\varphi}_\rho(\tau)$ with $\tilde{\varphi}_\rho(\beta) = \tilde{\varphi}_\rho(0)$. Substitution of this into eq. (33) yields

$$\bar{Z}_\pm = \sum_{m=-\infty}^{\infty} \exp\left(-\frac{\pi L}{\beta v_\rho K_\rho} m^2 + im(\chi + \zeta_\pm)\right). \quad (34)$$

Applying Poisson's summation theorem to eq. (34) and then substituting the resulting expression into eq. (32), we obtain the correct partition function in the absence

of the barriers,²¹

$$\bar{Z} \propto \sum_{\substack{J, M \\ (J+M=\text{even})}} \exp \left(-\beta \frac{\pi K_\rho v_\rho}{4L} \left(J + \frac{\chi}{\pi} \right)^2 - \beta \frac{\pi v_\sigma}{4K_\sigma L} M^2 \right), \quad (35)$$

where the summation over J and M is carried out under the condition of $J + M = \text{even}$.

Now we turn to the strong-barrier limit, where H^B is regarded as a strong pinning potential for the phase fields $\bar{\theta}$, $\tilde{\theta}$, $\bar{\phi}$ and $\tilde{\phi}$. It is convenient to introduce the effective action which satisfies

$$\frac{\int \prod_{q>0} \mathcal{D}a_q^\dagger \mathcal{D}a_q \mathcal{D}b_q^\dagger \mathcal{D}b_q \exp(-S^{\text{NZ}} - S^B)}{\int \prod_{q>0} \mathcal{D}a_q^\dagger \mathcal{D}a_q \mathcal{D}b_q^\dagger \mathcal{D}b_q \exp(-S^{\text{NZ}})} = \frac{\int \mathcal{D}\bar{\theta} \mathcal{D}\tilde{\theta} \mathcal{D}\bar{\phi} \mathcal{D}\tilde{\phi} \exp(-S_{\text{eff}}^{\text{NZ}} - S^B)}{\int \mathcal{D}\bar{\theta} \mathcal{D}\tilde{\theta} \mathcal{D}\bar{\phi} \mathcal{D}\tilde{\phi} \exp(-S_{\text{eff}}^{\text{NZ}})}. \quad (36)$$

This action is given by

$$\begin{aligned} S_{\text{eff}}^{\text{NZ}} = & \frac{1}{2\pi K_\rho \beta} \sum_\omega \bar{J}_\rho^{-1}(\omega) \bar{\theta}(\omega) \bar{\theta}(-\omega) \\ & + \frac{1}{8\pi K_\rho \beta} \sum_\omega \tilde{J}_\rho^{-1}(\omega) \tilde{\theta}(\omega) \tilde{\theta}(-\omega) \\ & + \frac{1}{2\pi K_\sigma \beta} \sum_\omega \bar{J}_\sigma^{-1}(\omega) \bar{\phi}(\omega) \bar{\phi}(-\omega) \\ & + \frac{1}{8\pi K_\sigma \beta} \sum_\omega \tilde{J}_\sigma^{-1}(\omega) \tilde{\phi}(\omega) \tilde{\phi}(-\omega), \end{aligned} \quad (37)$$

where

$$\begin{aligned} \bar{J}_\rho(\omega) = & \frac{1}{2\omega \sinh\left(\frac{L\omega}{v_\rho}\right)} \left\{ \cosh\left(\frac{L\omega}{v_\rho}\right) + \cosh\left(\frac{(L-\xi)\omega}{v_\rho}\right) \right. \\ & \left. + \cosh\left(\frac{\xi\omega}{v_\rho}\right) + 1 \right\} - \frac{2v_\rho}{L\omega^2}, \end{aligned} \quad (38)$$

$$\begin{aligned} \tilde{J}_\rho(\omega) = & \frac{1}{2\omega \sinh\left(\frac{L\omega}{v_\rho}\right)} \left\{ \cosh\left(\frac{L\omega}{v_\rho}\right) + \cosh\left(\frac{(L-\xi)\omega}{v_\rho}\right) \right. \\ & \left. - \cosh\left(\frac{\xi\omega}{v_\rho}\right) - 1 \right\}, \end{aligned} \quad (39)$$

$$\begin{aligned} \bar{J}_\sigma(\omega) = & \frac{1}{2\omega \sinh\left(\frac{L\omega}{v_\sigma}\right)} \left\{ \cosh\left(\frac{L\omega}{v_\sigma}\right) - \cosh\left(\frac{(L-\xi)\omega}{v_\sigma}\right) \right. \\ & \left. - \cosh\left(\frac{\xi\omega}{v_\sigma}\right) + 1 \right\}, \end{aligned} \quad (40)$$

$$\begin{aligned} \tilde{J}_\sigma(\omega) = & \frac{1}{2\omega \sinh\left(\frac{L\omega}{v_\sigma}\right)} \left\{ \cosh\left(\frac{L\omega}{v_\sigma}\right) - \cosh\left(\frac{(L-\xi)\omega}{v_\sigma}\right) \right. \\ & \left. + \cosh\left(\frac{\xi\omega}{v_\sigma}\right) - 1 \right\}. \end{aligned} \quad (41)$$

The derivation of the effective action is outlined in Appendix B. From eqs. (40) and (41), we see that $\bar{J}_\sigma^{-1}(\omega)$

and $\tilde{J}_\sigma^{-1}(\omega)$ are of the order of Δ in the limit of $\omega \rightarrow 0$. This indicates that $\bar{\phi}$ and $\tilde{\phi}$ have a mass gap of the order of Δ and thereby their low-energy fluctuations are strongly suppressed. We thus see that $\bar{\phi}$ and $\tilde{\phi}$ are pinned by H^B when $T \ll \Delta$. This is an important feature of the phase variables in our problem. We can understand the feature by noting that only a spin-singlet pair of two electrons can transfer the NS interfaces due to Andreev reflection. Detailed discussion on the characteristic features of $S_{\text{eff}}^{\text{NZ}}$ is given in §5.

Since $\bar{\phi}$ and $\tilde{\phi}$ are pinned by the barrier potential, we can set $\bar{\phi} = \tilde{\phi} = 0$ without loss of generality. Furthermore, we note that M plays only a minor role in changing the location of the potential minima, at which $\varphi + \bar{\theta}$ and $\tilde{\varphi} + \tilde{\theta}$ are pinned. Consequently, S^B can be simplified to

$$S^B = -\frac{4V_0}{\pi\alpha} \int_0^\beta d\tau \cos(\varphi_\rho + \bar{\theta}) \cos \frac{\tilde{\theta}}{2}, \quad (42)$$

where $k_F L = 0 \pmod{\pi}$ is assumed for simplicity. Since $\bar{\phi}$ and $\tilde{\phi}$ are irrelevant for our argument, $S_{\text{eff}}^{\text{NZ}}$ is reduced to

$$\begin{aligned} S_{\text{eff}}^{\text{NZ}} = & \frac{1}{2\pi K_\rho \beta} \sum_\omega \bar{J}_\rho^{-1}(\omega) \bar{\theta}(\omega) \bar{\theta}(-\omega) \\ & + \frac{1}{8\pi K_\rho \beta} \sum_\omega \tilde{J}_\rho^{-1}(\omega) \tilde{\theta}(\omega) \tilde{\theta}(-\omega). \end{aligned} \quad (43)$$

It should be noted here that the high-frequency cutoff $\Lambda/2$ (2Λ) must be introduced for the summation in the first (second) term to avoid ultraviolet divergence (Λ is of the order of Δ). To do so, we introduce an additional action which serves as the high-frequency cutoff for $S_{\text{eff}}^{\text{NZ}}$,¹⁰

$$S^\Lambda = \int_0^\beta d\tau \left(\frac{\bar{M}}{2} (\partial_\tau \bar{\theta})^2 + \frac{\tilde{M}}{2} (\partial_\tau \tilde{\theta})^2 \right), \quad (44)$$

where $\bar{M} \approx 2/\Lambda$ and $\tilde{M} \approx 1/(2\Lambda)$. After these treatments, eq. (28) is reduced to

$$\begin{aligned} Z_\pm = & \sum_{m=-\infty}^{\infty} e^{im(\chi+\zeta_\pm)} \int_{\Delta\varphi_\rho=2\pi m} \mathcal{D}\varphi_\rho \int \mathcal{D}\bar{\theta} \mathcal{D}\tilde{\theta} \\ & \times \exp(-S^Z - S_{\text{eff}}^{\text{NZ}} - S^\Lambda - S^B). \end{aligned} \quad (45)$$

In the strong-barrier limit, the change in φ_ρ with $m \neq 0$ occurs in collaboration with instanton tunneling of $\bar{\theta}$ and $\tilde{\theta}$ from a potential minimum $(\varphi_\rho + \bar{\theta}, \tilde{\theta}) = (k\pi, 2l\pi)$ to an adjacent minimum $(\varphi_\rho + \bar{\theta}, \tilde{\theta}) = ((k \pm 1)\pi, (2l \pm 2)\pi)$, where k and l are integers. The matrix element γ for this tunneling process is very small, so that we are allowed to consider only the second-order processes with respect to γ . Thus, these lowest-order tunneling processes correspond to the change in φ_ρ with $m = \pm 1$. Within this lowest-order approximation, Z_\pm is simplified to

$$Z_\pm \propto 1 \pm 2 \cos \chi \cdot Z_1/Z_0, \quad (46)$$

where

$$\begin{aligned} Z_m = & \int_{\Delta\varphi_\rho=2\pi m} \mathcal{D}\varphi_\rho \int \mathcal{D}\bar{\theta} \mathcal{D}\tilde{\theta} \\ & \times \exp(-S^Z - S_{\text{eff}}^{\text{NZ}} - S^\Lambda - S^B). \end{aligned} \quad (47)$$

To calculate Z_1 , we first determine the stationary path of $S_{\text{st}} \equiv S^Z + S^A + S^B$ under the condition of $\varphi_\rho(\beta) = \varphi_\rho(0) + 2\pi$, and then incorporate the influence of $S_{\text{eff}}^{\text{NZ}}$, which plays the role of a dissipative environment, by integrating out the low-energy fluctuations around the stationary path. We introduce the new variables:

$$\theta_r = \varphi_\rho + \bar{\theta}, \quad (48)$$

$$\theta_R = \frac{1}{m_L + \bar{M}} (m_L \varphi_\rho - \bar{M} \bar{\theta}), \quad (49)$$

where $m_L = L/(2\pi v_\rho K_\rho)$. Note that θ_r (θ_R) is the relative (center-of-mass) coordinate with respect to φ_ρ and $-\bar{\theta}$. Then, S_{st} becomes

$$S_{\text{st}} = \int_0^\beta d\tau \left(\frac{M_r}{2} (\partial_\tau \theta_r)^2 + \frac{M_R}{2} (\partial_\tau \theta_R)^2 + \frac{\tilde{M}}{2} (\partial_\tau \tilde{\theta})^2 - \frac{4V_0}{\pi\alpha} \cos \theta_r \cos \frac{\tilde{\theta}}{2} \right), \quad (50)$$

where $M_r = m_L \bar{M}/(m_L + \bar{M})$ and $M_R = m_L + \bar{M}$. The stationary path is determined under the boundary condition of $\theta_R(\beta) = \theta_R(0) + 2\pi m_L/(m_L + \bar{M})$ and $\theta_r(\beta) = \theta_r(0) + 2\pi$. Noting that $m_L \gg \bar{M}$, we find that $\theta_R \approx \varphi_\rho$. Thus, we approximately obtain the stationary path as

$$\theta_R^{\text{st}} \approx \varphi_\rho^{\text{st}} = \frac{2\pi}{\beta} \tau, \quad (51)$$

$$\theta_r^{\text{st}} = \frac{1}{2} I(\tau - \tau_1) + \frac{1}{2} I(\tau - \tau_2), \quad (52)$$

$$\tilde{\theta}^{\text{st}} = I(\tau - \tau_1) - I(\tau - \tau_2), \quad (53)$$

where $I(\tau - \tau_i)$ describes one instanton at $\tau = \tau_i$ and satisfies $I(-\infty) = 0$ and $I(\infty) = 2\pi$. Then, we find that

$$\bar{\theta}^{\text{st}} = \frac{1}{2} I(\tau - \tau_1) + \frac{1}{2} I(\tau - \tau_2) - \frac{2\pi}{\beta} \tau. \quad (54)$$

We calculate Z_1 taking account of the low-energy fluctuations around the stationary path. Due to the barrier potential, $\bar{\theta}$ and $\varphi_\rho + \bar{\theta}$ are strongly pinned. Thus, we can neglect the fluctuations of $\tilde{\theta}$ around $\tilde{\theta}^{\text{st}}$. In contrast, φ_ρ and $\bar{\theta}$ can fluctuate under the condition that $\varphi_\rho + \bar{\theta}$ is strongly pinned. This means that they are expressed as

$$\varphi_\rho = \varphi_\rho^{\text{st}} + \phi, \quad (55)$$

$$\bar{\theta} = \bar{\theta}^{\text{st}} - \phi, \quad (56)$$

where ϕ represents the low-energy fluctuations. We approximate that $I(\tau) = 2\pi\vartheta(\tau)$, where $\vartheta(\tau)$ is the step function. The Fourier transform of $\bar{\theta}^{\text{st}}$ and that of $\tilde{\theta}^{\text{st}}$ are expressed as

$$\bar{\theta}^{\text{st}}(\omega) = \frac{i\pi}{\omega} e^{i\omega\tau_1} + \frac{i\pi}{\omega} e^{i\omega\tau_2}, \quad (57)$$

$$\tilde{\theta}^{\text{st}}(\omega) = \frac{i2\pi}{\omega} e^{i\omega\tau_1} - \frac{i2\pi}{\omega} e^{i\omega\tau_2}. \quad (58)$$

Substituting eqs. (57) and (58) into eq. (47), we find that

$$\frac{Z_1}{Z_0} = \gamma^2 \int_0^\beta d\tau_1 d\tau_2 \int \mathcal{D}\phi e^{-S_{\text{ins}}}, \quad (59)$$

where

$$S_{\text{ins}} = \frac{\pi L}{v_\rho K_\rho \beta} + \frac{1}{\beta} \sum_{\omega>0} \frac{L\omega^2}{2\pi v_\rho K_\rho} \phi(\omega) \phi(-\omega) + \frac{1}{\pi K_\rho \beta} \sum_{\omega>0} \bar{J}_\rho^{-1}(\omega) (\bar{\theta}^{\text{st}}(\omega) - \phi(\omega)) \times (\bar{\theta}^{\text{st}}(-\omega) - \phi(-\omega)) + \frac{1}{4\pi K_\rho \beta} \sum_{\omega>0} \tilde{J}_\rho^{-1}(\omega) \tilde{\theta}^{\text{st}}(\omega) \tilde{\theta}^{\text{st}}(-\omega). \quad (60)$$

Integrating out ϕ , we obtain

$$\frac{Z_1}{Z_0} = \gamma^2 \exp\left(-\frac{\pi L}{K_\rho v_\rho} T\right) \int_0^\beta d\tau_1 d\tau_2 e^{-\tilde{S}_{\text{ins}}}, \quad (61)$$

where

$$\tilde{S}_{\text{ins}} = \frac{4\pi}{K_\rho \beta} \sum_{\omega>0} \frac{1}{\frac{2v_\rho}{L} + \omega^2 \bar{J}_\rho(\omega)} + \frac{2\pi}{K_\rho \beta} \sum_{\omega>0} \left(\frac{1}{\omega^2 \tilde{J}_\rho(\omega)} - \frac{1}{\frac{2v_\rho}{L} + \omega^2 \tilde{J}_\rho(\omega)} \right) \times (1 - \cos \omega(\tau_1 - \tau_2)). \quad (62)$$

We substitute eqs. (38) and (39) into eq. (62). Noting that $L \gg \xi$, we obtain

$$\tilde{S}_{\text{ins}} = \frac{4\pi}{K_\rho \beta} \sum_{\Lambda>\omega>0} \frac{1}{\omega} \tanh\left(\frac{L\omega}{2v_\rho}\right) + \frac{4\pi}{K_\rho \beta} \sum_{\omega>0} \frac{1}{\omega} \text{cosech}\left(\frac{L\omega}{v_\rho}\right) (1 - \cos \omega(\tau_1 - \tau_2)), \quad (63)$$

where we have introduced the high-frequency cutoff Λ in the first term.

We first consider the high-temperature regime of $v_F/L \ll T \ll \Delta$. Noting that

$$\sum_{\substack{M=-\infty \\ (M=\text{even})}}^{\infty} \exp\left(-\beta \frac{\pi v_\sigma}{4K_\sigma L} M^2\right) \propto 1 + 2 \exp\left(-\frac{\pi K_\sigma L}{v_\sigma} T\right), \quad (64)$$

$$\sum_{\substack{M=-\infty \\ (M=\text{odd})}}^{\infty} \exp\left(-\beta \frac{\pi v_\sigma}{4K_\sigma L} M^2\right) \propto 1 - 2 \exp\left(-\frac{\pi K_\sigma L}{v_\sigma} T\right), \quad (65)$$

the partition function is expressed as

$$Z \propto 2 + 8 \cos \chi \cdot \gamma^2 \exp\left(-\pi \left(\frac{L}{v_\rho K_\rho} + \frac{K_\sigma L}{v_\sigma}\right) T\right) \times \int_0^\beta d\tau_1 d\tau_2 e^{-\tilde{S}_{\text{ins}}}. \quad (66)$$

In the high-temperature regime, \tilde{S}_{ins} is obtained as

$$\tilde{S}_{\text{ins}} = \frac{2}{K_\rho} \ln\left(\frac{\Lambda}{2\pi T}\right). \quad (67)$$

Substitution of eq. (67) into eq. (66) yields

$$Z \propto 2 + 8 \cos \chi \cdot \gamma^2 \beta^2 \left(\frac{2\pi T}{\Lambda} \right)^{\frac{2}{\kappa_\rho}} \times \exp \left(-\pi \left(\frac{L}{v_\rho K_\rho} + \frac{K_\sigma L}{v_\sigma} \right) T \right). \quad (68)$$

From eq. (26), we finally obtain

$$j(\chi) = 8eT \left(\frac{2\pi\gamma}{\Lambda} \right)^2 \left(\frac{2\pi T}{\Lambda} \right)^{2(\frac{1}{\kappa_\rho}-1)} \times \exp \left(-\pi \left(\frac{L}{v_\rho K_\rho} + \frac{K_\sigma L}{v_\sigma} \right) T \right) \sin \chi. \quad (69)$$

Next, we turn to the low-temperature regime of $T \ll v_F/L$. In this regime, we are allowed to retain only the term with $M = 0$ in eq. (27). Thus, the partition function is given by

$$Z \propto 1 + 2 \cos \chi \cdot \gamma^2 \int_0^\beta d\tau_1 d\tau_2 e^{-\tilde{S}_{\text{ins}}}. \quad (70)$$

To evaluate \tilde{S}_{ins} , we approximate as

$$\frac{4\pi}{K_\rho \beta} \sum_{\Lambda > \omega > 0} \frac{1}{\omega} \tanh \left(\frac{L\omega}{2v_\rho} \right) \approx \frac{2}{K_\rho} \ln \left(\frac{2e^\gamma}{\pi} \cdot \frac{\Lambda L}{v_\rho} \right), \quad (71)$$

where $\gamma \approx 0.5772$. The second term in \tilde{S}_{ins} is obtained as

$$\begin{aligned} \frac{4\pi}{K_\rho \beta} \sum_{\omega > 0} \frac{1}{\omega} \text{cosech} \left(\frac{L\omega}{v_\rho} \right) (1 - \cos \omega (\tau_1 - \tau_2)) \\ \approx \frac{2}{K_\rho} \ln \left[\cosh \left(\frac{\pi v_\rho}{2L} (\tau_1 - \tau_2) \right) \right]. \end{aligned} \quad (72)$$

Using eqs. (71) and (72), we obtain

$$\int_0^\beta d\tau_1 d\tau_2 e^{-\tilde{S}_{\text{ins}}} = c \frac{\beta}{\Lambda} \left(\frac{v_\rho}{\Lambda L} \right)^{\frac{2}{\kappa_\rho}-1}, \quad (73)$$

where c is a numerical constant of order of unity. Substitution of eq. (73) into eq. (70) yields

$$Z \propto 1 + 2 \cos \chi \cdot c \frac{v_\rho \beta}{L} \left(\frac{\gamma}{\Lambda} \right)^2 \left(\frac{v_\rho}{\Lambda L} \right)^{2(\frac{1}{\kappa_\rho}-1)}. \quad (74)$$

We finally obtain

$$j(\chi) = 4c \frac{ev_\rho}{L} \left(\frac{\gamma}{\Lambda} \right)^2 \left(\frac{v_\rho}{\Lambda L} \right)^{2(\frac{1}{\kappa_\rho}-1)} \sin \chi. \quad (75)$$

This indicates that the critical current in the low-temperature regime behaves like $j_c \propto (1/L)^{2\kappa_\rho-1}$. This result is not consistent with Fazio *et al.*'s result, $j_c^{\text{FHO}} \propto (1/L)^{K_\rho-1+K_\sigma-1}$. We discuss the reason of this inconsistency in the final section.

4. Josephson Current in the Weak-Coupling Model

In this section we calculate the dc Josephson current in a TL liquid of length L with open boundaries, which is weakly coupled with the left (right) superconductor at $x = 0$ ($x = L$) through a tunnel junction (see Fig. 1(a)). We show that the resulting expression of the Josephson current is essentially equivalent to that in the interface-barrier model studied in §3.

The bosonized description of a TL liquid with open boundaries is presented by Fabrizio and Gogolin.²⁹ In terms of the boson operators a_q , a_q^\dagger , b_q and b_q^\dagger , and the winding-number operators N and M , the Hamiltonian is given by

$$H = \frac{\pi}{4L} \left(\frac{v_\rho}{K_\rho} N^2 + \frac{v_\sigma}{K_\sigma} M^2 \right) + \sum_{q>0} (v_\rho q a_q^\dagger a_q + v_\sigma q b_q^\dagger b_q), \quad (76)$$

where $q = \pi n/L$ ($n = 1, 2, 3, \dots$). The winding-number operators satisfy the condition of $N + M = \text{even}$. The phase fields are given by

$$\Theta_+(x) = \frac{\pi N}{L} x + \theta_+(x), \quad (77)$$

$$\Theta_-(x) = \varphi_\rho + \theta_-(x), \quad (78)$$

$$\Phi_+(x) = \frac{\pi M}{L} x + \phi_+(x), \quad (79)$$

$$\Phi_-(x) = \varphi_\sigma + \phi_-(x), \quad (80)$$

where θ_\pm and ϕ_\pm are the nonzero-mode components, and φ_ρ and φ_σ are the zero-mode operators which satisfy $[N, \varphi_\rho] = [M, \varphi_\sigma] = 2i$ and $[N, \varphi_\sigma] = [M, \varphi_\rho] = 0$. The nonzero-mode components are given by

$$\theta_+(x) = \sqrt{K_\rho} \sum_{q>0} \sqrt{\frac{2\pi}{qL}} e^{-\alpha q/2} \sin(qx) (a_q^\dagger + a_q), \quad (81)$$

$$\theta_-(x) = \frac{i}{\sqrt{K_\rho}} \sum_{q>0} \sqrt{\frac{2\pi}{qL}} e^{-\alpha q/2} \cos(qx) (a_q^\dagger - a_q), \quad (82)$$

$$\phi_+(x) = \sqrt{K_\sigma} \sum_{q>0} \sqrt{\frac{2\pi}{qL}} e^{-\alpha q/2} \sin(qx) (b_q^\dagger + b_q), \quad (83)$$

$$\phi_-(x) = \frac{i}{\sqrt{K_\sigma}} \sum_{q>0} \sqrt{\frac{2\pi}{qL}} e^{-\alpha q/2} \cos(qx) (b_q^\dagger - b_q). \quad (84)$$

It is clear that $\sin(qx) = 0$ when $x = 0$ and L , so that $\theta_+(x)$ and $\phi_+(x)$ are pinned at the ends of the 1D system. That is, $\theta_+(0) = \theta_+(L) = 0$ and $\phi_+(0) = \phi_+(L) = 0$. This means that fluctuations in both charge and spin are strongly suppressed in the vicinity of both the ends due to the open boundary condition. This result plays an important role in our argument. In terms of the phase fields, the right-moving component ψ_{1s} and the left-moving component ψ_{2s} of the electron field are expressed as²⁹

$$\psi_{1s}(x) = \frac{1}{\sqrt{2\pi\alpha}} \exp \left(ik_F x + \frac{i}{2} \{ \Theta_+(x) + \Theta_-(x) + s\Phi_+(x) + s\Phi_-(x) \} \right), \quad (85)$$

$$\psi_{2s}(x) = \frac{-1}{\sqrt{2\pi\alpha}} \exp \left(-ik_F x + \frac{i}{2} \{ -\Theta_+(x) + \Theta_-(x) - s\Phi_+(x) + s\Phi_-(x) \} \right). \quad (86)$$

We neglected Majorana fermions in eqs. (85) and (86), which do not play an essential role in our argument.

Now we introduce the Hamiltonian which describes the superconductors and their coupling with the TL liquid. We assume that our STLLS system is symmetric with respect to the NS contacts. That is, the left contact at

$x = 0$ is equivalent to the right contact at $x = L$. Thus, we present here only the part of the Hamiltonian related with the left superconductor. The Hamiltonian for the left superconductor is given by

$$H^S = \sum_{k,s} \epsilon_k c_{Lk,s}^\dagger c_{Lk,s} - \sum_k \left(\Delta e^{i\chi_1} c_{Lk,+}^\dagger c_{L-k,-}^\dagger + \Delta e^{-i\chi_1} c_{L-k,-} c_{Lk,+} \right), \quad (87)$$

where ϵ_k denotes the single electron spectrum and $c_{Lk,s}$ ($c_{Lk,s}^\dagger$) is the fermion annihilation (creation) operator. In terms of the Bogoliubov transformation

$$c_{Lk,+} = u_k d_{Lk,+} + v_k e^{i\chi_1} d_{L-k,-}^\dagger, \quad (88)$$

$$c_{L-k,-} = u_k d_{L-k,-} - v_k e^{i\chi_1} d_{Lk,+}^\dagger, \quad (89)$$

with

$$u_k = \sqrt{\frac{1}{2} \left(1 + \frac{\epsilon_k}{E_k} \right)}, \quad (90)$$

$$v_k = \sqrt{\frac{1}{2} \left(1 - \frac{\epsilon_k}{E_k} \right)}, \quad (91)$$

the Hamiltonian is diagonalized as

$$H^S = \sum_{k,s} E_k d_{Lk,s}^\dagger d_{Lk,s}, \quad (92)$$

where $E_k = \sqrt{\epsilon_k^2 + \Delta^2}$. The weak coupling between the TL liquid and the left superconductor is described by the tunneling Hamiltonian

$$H^T = \frac{1}{\sqrt{V}} \sum_{k,s} \left(c_{Lk,s}^\dagger \int dx t_{Lk}(x) \psi_s(x) + \text{h.c.} \right), \quad (93)$$

where V is the volume of the superconductor. We assume that the tunneling-matrix element $t_{Lk}(x)$ has nonzero values only in the vicinity of $x = 0$ and vanishes when $\alpha_T < x$, where α_T represents the length scale of the order of a few lattice spacings. The Hamiltonian for the right superconductor is simply given by replacing L with R in eqs. (92) and (93). Note that $t_{Rk}(x)$ has nonzero values only in the vicinity of $x = L$ and vanishes when $x < L - \alpha_T$.

In order to calculate the partition function, it is convenient to derive the effective action S^Γ describing the coupling with the TL liquid and the superconductors. The action S_Γ is obtained by integrating out the electron fields in the superconductors,

$$\int \prod_{k,s} \mathcal{D} d_{Lk,s}^\dagger \mathcal{D} d_{Lk,s} \mathcal{D} d_{Rk,s}^\dagger \mathcal{D} d_{Rk,s} \exp(-S^S - S^T) \propto \exp(-S^\Gamma), \quad (94)$$

where

$$S^S = \int_0^\beta d\tau \sum_{k,s} \left(d_{Lk,s}^\dagger \partial_\tau d_{Lk,s} + E_k d_{Lk,s}^\dagger d_{Lk,s} + d_{Rk,s}^\dagger \partial_\tau d_{Rk,s} + E_k d_{Rk,s}^\dagger d_{Rk,s} \right), \quad (95)$$

$$S^T = \int_0^\beta d\tau \frac{1}{\sqrt{V}} \sum_{k,s} \left(c_{Lk,s}^\dagger \int dx t_{Lk}(x) \psi_s(x) + c_{Rk,s}^\dagger \int dx t_{Rk}(x) \psi_s(x) + \text{h.c.} \right). \quad (96)$$

The derivation of S^Γ is outlined in Appendix C. The result is

$$S^\Gamma = \int_0^\beta d\tau_1 d\tau_2 Q(\tau_1 - \tau_2) \times \left(e^{-i\chi_1} \{ \psi_{1-}(0, \tau_1) \psi_{2+}(0, \tau_2) + \psi_{2-}(0, \tau_1) \psi_{1+}(0, \tau_2) \} + \text{h.c.} + e^{-i\chi_2} \{ \psi_{1-}(L, \tau_1) \psi_{2+}(L, \tau_2) + \psi_{2-}(L, \tau_1) \psi_{1+}(L, \tau_2) \} + \text{h.c.} \right), \quad (97)$$

where $Q(\tau)$ is defined in Appendix C. We substitute eqs. (85) and (86) into eq. (97). Noting that $\theta_+(0) = \theta_+(L) = 0$ and $\phi_+(0) = \phi_+(L) = 0$, we obtain

$$S^\Gamma = -\frac{1}{\pi\alpha} \int_0^\beta d\tau_1 d\tau_2 Q(\tau_1 - \tau_2) \times \left[e^{-i\chi_1} \exp \frac{i}{2} \left(\Theta_-(0, \tau_1) + \Theta_-(0, \tau_2) - \Phi_-(0, \tau_1) + \Phi_-(0, \tau_2) \right) + \text{c.c.} + (-1)^M e^{-i\chi_2} \exp \frac{i}{2} \left(\Theta_-(L, \tau_1) + \Theta_-(L, \tau_2) - \Phi_-(L, \tau_1) + \Phi_-(L, \tau_2) \right) + \text{c.c.} \right]. \quad (98)$$

The kernel $Q(\tau)$ vanishes when $|\tau| \gg \Delta^{-1}$ as shown in Appendix C, while the characteristic time scale of Θ_+ and Φ_- is much longer than Δ^{-1} . Thus, we can approximate in eq. (98) that

$$Q(\tau_i - \tau_j) = \Gamma \delta(\tau_i - \tau_j), \quad (99)$$

where $\Gamma = \int d\tau Q(\tau)$. Thus, S^Γ is simplified to

$$S^\Gamma = -\frac{\Gamma}{\pi\alpha} \int_0^\beta d\tau \left[\left(e^{-i\chi_1 + i\Theta_-(0,\tau)} + \text{c.c.} \right) + (-1)^M \left(e^{-i\chi_2 + i\Theta_-(L,\tau)} + \text{c.c.} \right) \right]. \quad (100)$$

The sign of the second term depends on M , so we express S^Γ with even M and that of odd M as S_+^Γ and S_-^Γ , respectively.

Note that S_\pm^Γ does not contain φ_σ , so that M is a conserved quantity. By contrast, N is no longer conserved in the presence of S_\pm^Γ . The partition function is expressed

as

$$Z = \sum_{\substack{M=-\infty \\ (M=\text{even})}}^{\infty} \exp\left(-\beta \frac{\pi v_{\sigma} M^2}{4K_{\sigma} L}\right) Z_+ \\ + \sum_{\substack{M=-\infty \\ (M=\text{odd})}}^{\infty} \exp\left(-\beta \frac{\pi v_{\sigma} M^2}{4K_{\sigma} L}\right) Z_-, \quad (101)$$

with

$$Z_{\pm} = \sum_{m=-\infty}^{\infty} \int_{\Delta\varphi_{\rho}=2\pi m} \mathcal{D}\varphi_{\rho} \int \prod_{q>0} \mathcal{D}a_q^{\dagger} \mathcal{D}a_q \\ \times \exp\left(i \int_0^{\beta} d\tau \frac{\zeta_{\pm}}{2\pi} \partial_{\tau} \varphi_{\rho} - S^Z - S^{\text{NZ}} - S_{\pm}^{\Gamma}\right), \quad (102)$$

where $\zeta_+ = 0$ and $\zeta_- = \pi$, and

$$S^Z = \int_0^{\beta} d\tau \frac{L}{4\pi v_{\rho} K_{\rho}} (\partial_{\tau} \varphi_{\rho})^2, \quad (103)$$

$$S^{\text{NZ}} = \int_0^{\beta} d\tau \sum_{q>0} (a_q^{\dagger} \partial_{\tau} a_q + v_{\rho} q a_q^{\dagger} a_q). \quad (104)$$

We introduced ζ_{\pm} to maintain the constraint $N + M = \text{even}$.³³ Since it is assumed that the coupling between the TL liquid and the superconductors is weak, we calculate the partition function by treating S_{\pm}^{Γ} as a perturbation. Within the lowest-order approximation with respect to S_{\pm}^{Γ} , we find that $Z_{\pm} \propto 1 + Z_{1\pm}/Z_{0\pm}$, where

$$Z_{0\pm} = \sum_{m=-\infty}^{\infty} \int_{\Delta\varphi_{\rho}=2\pi m} \mathcal{D}\varphi_{\rho} \int \prod_{q>0} \mathcal{D}a_q^{\dagger} \mathcal{D}a_q \\ \times \exp(im\zeta_{\pm} - S^Z - S^{\text{NZ}}), \quad (105)$$

$$Z_{1\pm} = \sum_{m=-\infty}^{\infty} \int_{\Delta\varphi_{\rho}=2\pi m} \mathcal{D}\varphi_{\rho} \int \prod_{q>0} \mathcal{D}a_q^{\dagger} \mathcal{D}a_q \frac{1}{2} (S_{\pm}^{\Gamma})^2 \\ \times \exp(im\zeta_{\pm} - S^Z - S^{\text{NZ}}). \quad (106)$$

Using eq. (100), we find that

$$Z_{1\pm} = \pm \sum_{m=-\infty}^{\infty} e^{im\zeta_{\pm}} \int_{\Delta\varphi_{\rho}=2\pi m} \mathcal{D}\varphi_{\rho} \int \prod_{q>0} \mathcal{D}a_q^{\dagger} \mathcal{D}a_q \\ \times \exp(-S^Z - S^{\text{NZ}}) \\ \times \left(\frac{\Gamma}{\pi\alpha}\right)^2 \int_0^{\beta} d\tau_1 d\tau_2 \\ \times \left(e^{i\chi} \exp i(\Theta_-(0, \tau_1) - \Theta_-(L, \tau_2)) + \text{c.c.}\right), \quad (107)$$

where $\chi = \chi_2 - \chi_1$ and we neglected terms which are independent of χ . To proceed, we decompose φ_{ρ} as $\varphi_{\rho}(\tau) = \varphi_{\rho}^0 + 2\pi m\tau/\beta + \tilde{\varphi}_{\rho}(\tau)$, where $\tilde{\varphi}_{\rho}(\beta) - \tilde{\varphi}_{\rho}(0) = 0$.

Using this decomposition, we obtain

$$Z_{0\pm} = \sum_{m=-\infty}^{\infty} e^{-\frac{\pi K_{\rho} L}{v_{\rho} \beta} m^2} \int \mathcal{D}\tilde{\varphi}_{\rho} \int \prod_{q>0} \mathcal{D}a_q^{\dagger} \mathcal{D}a_q \\ \times \exp(-S^Z - S^{\text{NZ}}), \quad (108)$$

$$Z_{1\pm} = \pm \left(\frac{\Gamma}{\pi\alpha}\right)^2 \int_0^{\beta} d\tau_1 d\tau_2 \sum_{m=-\infty}^{\infty} \\ \times e^{-\frac{\pi K_{\rho} L}{v_{\rho} \beta} m^2} e^{im\zeta_{\pm} + i\frac{2\pi m}{\beta}(\tau_1 - \tau_2)} \\ \times \int \mathcal{D}\tilde{\varphi}_{\rho} \int \prod_{q>0} \mathcal{D}a_q^{\dagger} \mathcal{D}a_q \exp(-S^Z - S^{\text{NZ}}) \\ \times \left\{ e^{i\chi} \exp i(\tilde{\varphi}_{\rho}(\tau_1) - \tilde{\varphi}_{\rho}(\tau_2) + \theta_-(0, \tau_1) \right. \\ \left. - \theta_-(L, \tau_2)) + \text{c.c.} \right\}. \quad (109)$$

To evaluate $Z_{1\pm}$, it is convenient to introduce the correlation function defined by

$$\Omega(\tau_1 - \tau_2) = \left\langle \exp i(\tilde{\varphi}_{\rho}(\tau_1) - \tilde{\varphi}_{\rho}(\tau_2) \right. \\ \left. + \theta_-(0, \tau_1) - \theta_-(L, \tau_2)) \right\rangle, \quad (110)$$

where

$$\langle \dots \rangle = \frac{\int \mathcal{D}\tilde{\varphi}_{\rho} \int \prod_{q>0} \mathcal{D}a_q^{\dagger} \mathcal{D}a_q \dots \exp(-S^Z - S^{\text{NZ}})}{\int \mathcal{D}\tilde{\varphi}_{\rho} \int \prod_{q>0} \mathcal{D}a_q^{\dagger} \mathcal{D}a_q \exp(-S^Z - S^{\text{NZ}})}. \quad (111)$$

The correlation function is obtained as

$$\Omega(\tau) = \exp\left(-\frac{\pi L}{K_{\rho} v_{\rho} \beta} - \frac{4\pi}{K_{\rho} \beta} \sum_{\Lambda>\omega>0} \frac{1}{\omega} \tanh\left(\frac{L\omega}{2v_{\rho}}\right) \right. \\ \left. - \frac{4\pi}{K_{\rho} \beta} \sum_{\omega>0} \frac{1}{\omega} \text{cosech}\left(\frac{L\omega}{v_{\rho}}\right) (1 - \cos \omega\tau) \right), \quad (112)$$

where the high-frequency cutoff Λ is introduced for the second term in the exponent. By using Ω , we can express $Z_{1\pm}/Z_{0\pm}$ as

$$\frac{Z_{1\pm}}{Z_{0\pm}} = \pm 2\gamma^2 \cos \chi \int_0^{\beta} d\tau_1 d\tau_2 \\ \times \frac{\sum_m e^{-\frac{\pi K_{\rho} L}{v_{\rho} \beta} m^2} e^{im\zeta_{\pm} + i\frac{2\pi m}{\beta}(\tau_1 - \tau_2)}}{\sum_m e^{-\frac{\pi K_{\rho} L}{v_{\rho} \beta} m^2}} \Omega(\tau_1 - \tau_2), \quad (113)$$

where we set $\gamma = \Gamma/(\pi\alpha)$.

We first calculate the Josephson current in the high-temperature regime of $v_{\rho}/L \ll T \ll \Delta$. In this regime, we are allowed to retain only the term with $m = 0$, and obtain

$$\frac{Z_{1\pm}}{Z_{0\pm}} = \pm 2\gamma^2 \cos \chi \int_0^{\beta} d\tau_1 d\tau_2 \Omega(\tau_1 - \tau_2), \quad (114)$$

where

$$\Omega(\tau) \approx \left(\frac{2\pi T}{\Lambda}\right)^{\frac{2}{K_{\rho}}} \exp\left(-\frac{\pi L}{K_{\rho} v_{\rho}} T\right). \quad (115)$$

The partition function is

$$\begin{aligned} Z &\propto \left(1 + 2e^{-\frac{\pi K_\sigma L}{v_\sigma T}}\right) \left(1 + \frac{Z_{1+}}{Z_{0+}}\right) \\ &\quad + \left(1 - 2e^{-\frac{\pi K_\sigma L}{v_\sigma T}}\right) \left(1 - \frac{Z_{1+}}{Z_{0+}}\right) \\ &= 2 + 8 \cos \chi \cdot \gamma^2 \beta^2 \left(\frac{2\pi T}{\Lambda}\right)^{\frac{2}{\kappa_\rho}} \\ &\quad \times \exp\left(-\pi \left(\frac{L}{K_\rho v_\rho} + \frac{K_\sigma L}{v_\sigma}\right) T\right). \end{aligned} \quad (116)$$

We have used eqs. (64) and (65) in deriving eq. (116). The Josephson current is obtained as

$$\begin{aligned} j(\chi) &= -2eT \frac{\partial \ln Z}{\partial \chi} \\ &= 8eT \left(\frac{2\pi\gamma}{\Lambda}\right)^2 \left(\frac{2\pi T}{\Lambda}\right)^{2\left(\frac{1}{\kappa_\rho}-1\right)} \\ &\quad \times \exp\left(-\pi \left(\frac{L}{K_\rho v_\rho} + \frac{K_\sigma L}{v_\sigma}\right) T\right) \sin \chi. \end{aligned} \quad (117)$$

This result is equivalent to that obtained in §3 in the high-temperature regime.

Next we consider the low-temperature regime of $T \ll v_\rho/L$, where the terms with $M = 0$ dominantly contribute to Z . We then find that $Z \propto 1 + Z_{1+}/Z_{0+}$. Using Poisson's summation theorem, we rewrite the summation over m in Z_{1+} ,

$$\begin{aligned} &\sum_{m=-\infty}^{\infty} e^{-\frac{\pi K_\rho L}{v_\rho \beta} m^2} e^{i\frac{2\pi m}{\beta}(\tau_1 - \tau_2)} \\ &\propto \sum_{q=-\infty}^{\infty} \exp\left(-\frac{v_\rho \beta}{4\pi K_\rho L} \left(2\pi q + \frac{2\pi}{\beta}(\tau_1 - \tau_2)\right)^2\right). \end{aligned} \quad (118)$$

This equation indicates that we are allowed to retain only the term with $q = 0$. Then, we obtain

$$\begin{aligned} \frac{Z_{1+}}{Z_{0+}} &= 2 \cos \chi \cdot \gamma^2 \beta^2 \int_0^\beta d\tau_1 d\tau_2 e^{-\frac{\pi v_\rho}{K_\rho L \beta}(\tau_1 - \tau_2)^2} \\ &\quad \times \Omega(\tau_1 - \tau_2), \end{aligned} \quad (119)$$

with

$$\Omega(\tau) \approx c' \left(\frac{v_\rho}{\Lambda L}\right)^{\frac{2}{\kappa_\rho}} \cosh\left(\frac{\pi v_\rho}{2L} \tau\right)^{-\frac{2}{\kappa_\rho}}, \quad (120)$$

where c' is a numerical constant of order of unity. The partition function is obtained as

$$Z \propto 1 + 2 \cos \chi \cdot c \gamma^2 \frac{\beta}{\Lambda} \left(\frac{v_\rho}{\Lambda L}\right)^{\frac{2}{\kappa_\rho}-1}, \quad (121)$$

where c is a numerical constant of order of unity. We finally obtain

$$\begin{aligned} j(\chi) &= -2eT \frac{\partial \ln Z}{\partial \chi} \\ &= 4c \frac{ev_\rho}{L} \left(\frac{\gamma}{\Lambda}\right)^2 \left(\frac{v_\rho}{\Lambda L}\right)^{2\left(\frac{1}{\kappa_\rho}-1\right)} \sin \chi. \end{aligned} \quad (122)$$

The result is also equivalent to that obtained in §3 in the low-temperature regime.

5. Discussion

We studied the dc Josephson effect in a TL liquid on the basis of the two different models. We derived a bosonized description of a superconductor-TL-liquid-superconductor (STLLS) system with perfect NS interfaces, and calculate the Josephson current by introducing a strong barrier at each NS interface. We next examined the Josephson current through a TL liquid with open boundaries, where the TL liquid is weakly coupled at its left (right) end with the left (right) superconductor through a tunnel junction. It is shown that the two models provide us the same expression of the Josephson current in both the high-temperature regime of $v_F/L \ll T \ll \Delta$ and the low-temperature regime of $T \ll v_F/L$. We found that the Josephson current in the high-temperature regime is given by

$$\begin{aligned} j(\chi) &= 8eT \left(\frac{2\pi\gamma}{\Lambda}\right)^2 \left(\frac{2\pi T}{\Lambda}\right)^{2\left(\frac{1}{\kappa_\rho}-1\right)} \\ &\quad \times \exp\left(-\pi \left(\frac{L}{v_\rho K_\rho} + \frac{K_\sigma L}{v_\sigma}\right) T\right) \sin \chi, \end{aligned} \quad (123)$$

while

$$j(\chi) = 4c \frac{ev_\rho}{L} \left(\frac{\gamma}{\Lambda}\right)^2 \left(\frac{v_\rho}{\Lambda L}\right)^{2\left(\frac{1}{\kappa_\rho}-1\right)} \sin \chi, \quad (124)$$

in the low-temperature regime.

We treated the strong-barrier limit of the interface-barrier model in §3. Let us briefly consider the opposite weak-barrier case. For simplicity, both the barriers are assumed to have the same strength. We employ the notations presented in §3. The action describing the weak barrier potential is

$$S^B = -\frac{4V_0}{\pi\alpha} \int d\tau \cos(\varphi_\rho + \bar{\theta}) \cos \frac{\bar{\theta}}{2}, \quad (125)$$

with $V_0/(\alpha\Lambda) \ll 1$. We have used the fact that the spin degrees of freedom are frozen in the vicinity of the NS interfaces. The potential becomes large after renormalization. We apply a perturbative renormalization-group argument to obtain a scaling equation for the barrier potential.⁶⁻¹⁰ If the band width Λ is reduced to μ , the potential becomes

$$V_0(\mu) = V_0(\Lambda) \left(\frac{\mu}{\Lambda}\right)^{K_\rho-1}, \quad (126)$$

as long as $v_\rho/L \ll \mu < \Lambda$. This means that the potential is renormalized to be large in the repulsive interaction cases of $K_\rho < 1$. Thus, the system flows to the strong-barrier limit, where our analysis in §3 is justified. Since the coupling between a TL liquid and superconductors is reduced by the strong barriers in this limit, we expect that the weak-coupling model also describes correct low-energy physics. Our results given in §3 and §4 support this reasoning.

To elucidate characteristic features of the interface-barrier model, it is instructive to compare our problem with the double-barrier problem in a TL liquid. Kane and Fisher,⁷ and subsequently Furusaki and Nagaosa,⁹

studied transport properties of a TL liquid in the presence of a double-barrier structure, which consists of two δ -function barriers at $x = 0$ and L . This system is similar to our Josephson system except for the absence of superconductors. The Hamiltonian of the double-barrier problem is given by $H = H^{1D} + H^B$, where

$$H^{1D} = \int_{-\infty}^{\infty} dx \left(\frac{v_\rho}{4\pi K_\rho} \left(\frac{\partial \theta_+}{\partial x} \right)^2 + \frac{K_\rho v_\rho}{4\pi} \left(\frac{\partial \theta_-}{\partial x} \right)^2 \right) + \int_{-\infty}^{\infty} dx \left(\frac{v_\sigma}{4\pi K_\sigma} \left(\frac{\partial \phi_+}{\partial x} \right)^2 + \frac{K_\sigma v_\sigma}{4\pi} \left(\frac{\partial \phi_-}{\partial x} \right)^2 \right), \quad (127)$$

$$H^B = -\frac{4V_0}{\pi\alpha} \left[\cos \bar{\theta} \cos \frac{\tilde{\theta}}{2} \cos \bar{\phi} \cos \frac{\tilde{\phi}}{2} + \sin \bar{\theta} \sin \frac{\tilde{\theta}}{2} \sin \bar{\phi} \sin \frac{\tilde{\phi}}{2} \right]. \quad (128)$$

We assumed that $k_F L = 0 \pmod{\pi}$. The variables $\bar{\theta}$, $\tilde{\theta}$, $\bar{\phi}$ and $\tilde{\phi}$ in eq. (128) are defined as $\bar{\theta} = (\theta_+(L) + \theta_+(0))/2$, $\tilde{\theta} = \theta_+(L) - \theta_+(0)$, $\bar{\phi} = (\phi_+(L) + \phi_+(0))/2$ and $\tilde{\phi} = \phi_+(L) - \phi_+(0)$, respectively. Note that θ_- and ϕ_- are not contained in H^B . Thus, the partition function is

$$Z = \int \mathcal{D}\theta_+ \int \mathcal{D}\phi_+ \exp(-S^{1D} - S^B), \quad (129)$$

where

$$S^{1D} = \int_0^\beta d\tau \int_{-\infty}^{\infty} dx \left(\frac{v_\rho}{4\pi K_\rho} \left(\frac{\partial \theta_+}{\partial x} \right)^2 + \frac{1}{4\pi K_\rho v_\rho} \left(\frac{\partial \theta_+}{\partial \tau} \right)^2 \right) + \int_0^\beta d\tau \int_{-\infty}^{\infty} dx \left(\frac{v_\sigma}{4\pi K_\sigma} \left(\frac{\partial \phi_+}{\partial x} \right)^2 + \frac{1}{4\pi K_\sigma v_\sigma} \left(\frac{\partial \phi_+}{\partial \tau} \right)^2 \right), \quad (130)$$

$$S^B = \int_0^\beta d\tau H^B(\tau). \quad (131)$$

We can simplify the partition function by using the effective action for $\bar{\theta}$, $\tilde{\theta}$, $\bar{\phi}$ and $\tilde{\phi}$. The result is⁹

$$Z = \int \mathcal{D}\bar{\theta} \mathcal{D}\tilde{\theta} \int \mathcal{D}\bar{\phi} \mathcal{D}\tilde{\phi} \exp(-S_{\text{eff}}^{1D} - S^B), \quad (132)$$

where

$$S_{\text{eff}}^{1D} = \frac{1}{2\pi K_\rho \beta} \sum_\omega \frac{2|\omega|}{1 + \exp(-L|\omega|/v_\rho)} \bar{\theta}(\omega) \tilde{\theta}(-\omega) + \frac{1}{8\pi K_\rho \beta} \sum_\omega \frac{2|\omega|}{1 - \exp(-L|\omega|/v_\rho)} \tilde{\theta}(\omega) \tilde{\theta}(-\omega) + \frac{1}{2\pi K_\sigma \beta} \sum_\omega \frac{2|\omega|}{1 + \exp(-L|\omega|/v_\sigma)} \bar{\phi}(\omega) \tilde{\phi}(-\omega) + \frac{1}{8\pi K_\sigma \beta} \sum_\omega \frac{2|\omega|}{1 - \exp(-L|\omega|/v_\sigma)} \tilde{\phi}(\omega) \tilde{\phi}(-\omega). \quad (133)$$

We can clarify the influence of superconductors on a TL liquid by comparing the effective action $S_{\text{eff}}^{\text{NZ}}$ given in eq. (37) with that of the double-barrier problem S_{eff}^{1D} given in eq. (133). As noted in §3, we see from eqs. (40) and (41) that $\bar{J}_\sigma^{-1}(\omega)$ and $\tilde{J}_\sigma^{-1}(\omega)$ are of the order of Δ in the limit of $\omega \rightarrow 0$. This indicates that $\bar{\phi}$ and $\tilde{\phi}$ have a mass gap of the order of Δ in our problem, while such large gap does not appear in the double-barrier problem. We can understand the feature by noting that only a spin-singlet pair of two electrons can transfer the NS interfaces due to Andreev reflection. The conservation of the total spin through the Andreev reflection process leads to suppression of the spin fluctuations near the interfaces. The mass gap of the order of Δ reflects it. The presence of superconductors also affects the charge degrees of freedom. We also see that $\bar{J}_\rho^{-1}(\omega)$ and $\tilde{J}_\rho^{-1}(\omega)$ are of the order of v_ρ/L in the limit of $\omega \rightarrow 0$. Thus, $\bar{\theta}$ and $\tilde{\theta}$ have a small mass gap of the order of v_F/L . This gap is simply induced by a finite-size effect. Let us focus our attention on the regime of $v_\rho/L \ll |\omega|$, where the finite-size effect plays no role. The part of $S_{\text{eff}}^{\text{NZ}}$ describing the dynamics of $\bar{\theta}(\omega)$ with $v_\rho/L \ll |\omega|$ is given by

$$S_{\text{eff}}^{\text{NZ}}[\bar{\theta}] = \frac{1}{2\pi K_\rho \beta} \sum_{v_\rho/L \ll |\omega|} |\omega| \bar{\theta}(\omega) \bar{\theta}(-\omega). \quad (134)$$

Similarly, the part of S_{eff}^{1D} is

$$S_{\text{eff}}^{1D}[\bar{\theta}] = \frac{1}{2\pi K_\rho \beta} \sum_{v_\rho/L \ll |\omega|} 2|\omega| \bar{\theta}(\omega) \bar{\theta}(-\omega). \quad (135)$$

This indicates that the fluctuations in $\bar{\theta}$ are enhanced by a factor of two in our Josephson system compared with the double-barrier system. The fluctuations of $\tilde{\theta}$ are also enhanced by a factor of two. This feature is also attributed to Andreev reflection. The reason is that the unit of charge transfer is not e but $2e$ in the Andreev reflection process. This doubling of the unit charge enhances the charge fluctuations in the vicinity of the NS interfaces.

We obtained that the critical current is $j_c \propto (1/L)^{2K_\rho^{-1}-1}$ when $T \ll v_F/L$. As noted in §3, our result is not consistent with Fazio *et al.*'s result,^{19,20} $j_c^{\text{FHO}} \propto (1/L)^{K_\rho^{-1}+K_\sigma-1}$. The reason of this inconsistency is now clear. Fazio *et al.* treated a TL liquid of infinite length, which is weakly coupled with the left (right) superconductor at $x = 0$ ($x = L$), under the assumption that the potential induced by the coupling with the superconductors can be completely neglected. Thus, within their assumption, the dynamics of the phase fields in the 1D system is completely unaffected by the coupling. However, even a very weak potential becomes large after renormalization due to the repulsive electron-electron interactions in a TL liquid,⁶ so that the TL liquid is effectively disconnected at $x = 0$ and $x = L$. In such a situation, θ_+ and ϕ_+ are pinned at the disconnected points, while fluctuations in θ_- and ϕ_- are enhanced around there. From the above argument, we see that Fazio *et al.*'s result may apply to only restricted situations, where the induced potential is completely negligible. If the potential becomes very large after the renor-

malization, Fazio *et al.*'s model must be replaced with our weak-coupling model, as suggested by Fabrizio and Gogolin.²⁹ This means that our result is much general compared with Fazio *et al.*'s result.

Details of the difference between $j_c \propto (1/L)^{2K_\rho^{-1}-1}$ and $j_c^{\text{FHO}} \propto (1/L)^{K_\rho^{-1}+K_\sigma-1}$ are explained as follows. The dynamics of the phase fields is assumed to be unaffected by the coupling with the superconductors in refs. 19 and 20, while, in our weak-coupling model with the open-boundary condition, we explicitly incorporate the strong pinning of ϕ_+ and the enhanced low-energy fluctuations of θ_- in the vicinity of the NS interfaces. The correlation exponent K_σ does not appear in the expression of j_c due to the pinning of ϕ_+ , while j_c^{FHO} contains K_σ . The prefactor of K_ρ^{-1} in the exponent of j_c is doubled compared with j_c^{FHO} due to the enhancement in the fluctuations of θ_- .

Appendix A: Bosonization of TL liquid sandwiched between superconductors

In this appendix, we adapt the bosonization method to a TL liquid sandwiched between superconductors. We first bosonize the 1D electron system in the noninteracting limit, and then incorporate the influence of electron-electron interactions. Our bosonization procedure is based on the argument by Haldane.¹⁷

In terms of the eigenfunctions of the BdG equation, presented in eqs. (2) and (3), the electron-field operators defined in eq. (9) are expressed as³⁰

$$\psi_{1+}(x) = \sum_{q>0} \left(u_q(x) c_{q,+} - g_q^*(x) c_{q,-}^\dagger \right), \quad (\text{A}\cdot 1)$$

$$\psi_{2+}(x) = \sum_{q>0} \left(f_q(x) d_{q,+} - v_q^*(x) d_{q,-}^\dagger \right), \quad (\text{A}\cdot 2)$$

$$\psi_{1-}(x) = \sum_{q>0} \left(u_q(x) d_{q,-} + g_q^*(x) d_{q,+}^\dagger \right), \quad (\text{A}\cdot 3)$$

$$\psi_{2-}(x) = \sum_{q>0} \left(f_q(x) c_{q,-} - v_q^*(x) c_{q,+}^\dagger \right), \quad (\text{A}\cdot 4)$$

where $c_{q,s}$ and $d_{q,s}$ ($c_{q,s}^\dagger$ and $d_{q,s}^\dagger$) are the fermion annihilation (creation) operators and $q = \pi(n+1/2)/(L+\xi)$ with $n = 0, 1, 2, \dots$. The linearized quasiparticle dispersion is

$$\epsilon_{q\pm} = v_F \left(q \pm \frac{\chi}{2(L+\xi)} \right) + \frac{1}{2m} \left(\frac{\chi}{2(L+\xi)} \right)^2. \quad (\text{A}\cdot 5)$$

Thus, the quasiparticles in the 1D system are described by the Hamiltonian

$$\begin{aligned} H_0 = & \frac{v_F}{4\pi} \cdot \frac{\chi^2}{L+\xi} \\ & + \sum_{q>0} v_F \left(q + \frac{\chi}{2(L+\xi)} \right) \left(c_{q,+}^\dagger c_{q,+} + d_{q,-}^\dagger d_{q,-} \right) \\ & + \sum_{q>0} v_F \left(q - \frac{\chi}{2(L+\xi)} \right) \left(c_{q,-}^\dagger c_{q,-} + d_{q,+}^\dagger d_{q,+} \right). \end{aligned} \quad (\text{A}\cdot 6)$$

Here, the first term represents the χ -dependent component of the ground-state energy, arising from the second term in eq. (A·5).

We define the density operator $\rho_{js}(x)$ ($j = 1, 2$) as $\rho_{js}(x) = \psi_{js}^\dagger(x) \psi_{js}(x)$. Its Fourier transform is given by

$$\rho_{js}(q) = \int_0^L dx e^{iqx} \rho_{js}(x), \quad (\text{A}\cdot 7)$$

where $q = \pi(n+1/2)/(L+\xi)$ with $n = 0, \pm 1, \pm 2, \dots$. An important role in our bosonization procedure is played by the new density operators,

$$\rho_a(q) = e^{i\xi q/2} \rho_{1+}(q) - e^{-i\xi q/2} \rho_{2-}(-q), \quad (\text{A}\cdot 8)$$

$$\rho_b(q) = e^{i\xi q/2} \rho_{1-}(q) - e^{-i\xi q/2} \rho_{2+}(-q). \quad (\text{A}\cdot 9)$$

For $q > 0$, they are expressed as

$$\begin{aligned} \rho_a(q) = & \sum_{q_1>0} \left(c_{q_1+q,+}^\dagger c_{q_1,+} + c_{q_1,-} c_{q_1+q,-}^\dagger \right) \\ & - i \sum_{q>q_1>0} c_{-q_1+q,+}^\dagger c_{q_1,-}^\dagger, \end{aligned} \quad (\text{A}\cdot 10)$$

$$\begin{aligned} \rho_a(-q) = & \sum_{q_1>0} \left(c_{q_1,+}^\dagger c_{q_1+q,+} + c_{q_1+q,-} c_{q_1,-}^\dagger \right) \\ & + i \sum_{q>q_1>0} c_{q_1,-} c_{-q_1+q,+}, \end{aligned} \quad (\text{A}\cdot 11)$$

$$\begin{aligned} \rho_b(q) = & \sum_{q_1>0} \left(d_{q_1+q,-}^\dagger d_{q_1,-} + d_{q_1,+} d_{q_1+q,+}^\dagger \right) \\ & - i \sum_{q>q_1>0} d_{-q_1+q,-}^\dagger d_{q_1,+}^\dagger, \end{aligned} \quad (\text{A}\cdot 12)$$

$$\begin{aligned} \rho_b(-q) = & \sum_{q_1>0} \left(d_{q_1,-}^\dagger d_{q_1+q,-} + d_{q_1+q,+} d_{q_1,+}^\dagger \right) \\ & + i \sum_{q>q_1>0} d_{q_1,+} d_{-q_1+q,-}. \end{aligned} \quad (\text{A}\cdot 13)$$

We can show that

$$[\rho_a(q), \rho_a(-q')] = -\frac{Lq}{\pi} \delta_{q,q'}, \quad (\text{A}\cdot 14)$$

$$[\rho_b(q), \rho_b(-q')] = -\frac{Lq}{\pi} \delta_{q,q'}, \quad (\text{A}\cdot 15)$$

$$[\rho_a(q), \rho_b(-q')] = 0. \quad (\text{A}\cdot 16)$$

The $q = 0$ components, which characterize an excess charge with respect to the ground state, are given by

$$\rho_a(q=0) = N_a + \frac{\chi}{2\pi}, \quad (\text{A}\cdot 17)$$

$$\rho_b(q=0) = N_b + \frac{\chi}{2\pi}, \quad (\text{A}\cdot 18)$$

where $\chi/(2\pi)$ results from the shift of the occupied-state distribution due to the phase difference χ . We call N_a and N_b the winding-number operators, whose eigenvalues are $0, \pm 1, \pm 2, \dots$. We will employ the zero-mode operators φ_a and φ_b , which satisfy $[N_a, \varphi_a] = [N_b, \varphi_b] = i$ and $[N_a, \varphi_b] = [N_b, \varphi_a] = 0$.

Let us consider the commutation relations between the density operators with $q \neq 0$ and the field operators pre-

sented in eqs. (A·1)-(A·4). We find that

$$[\rho_a(q), \psi_{1+}(x)] = -e^{iq(x+\xi/2)}\psi_{1+}(x), \quad (\text{A}\cdot 19)$$

$$[\rho_a(q), \psi_{2-}(x)] = e^{-iq(x+\xi/2)}\psi_{2-}(x), \quad (\text{A}\cdot 20)$$

$$[\rho_b(q), \psi_{1-}(x)] = -e^{iq(x+\xi/2)}\psi_{1-}(x), \quad (\text{A}\cdot 21)$$

$$[\rho_b(q), \psi_{2+}(x)] = e^{-iq(x+\xi/2)}\psi_{2+}(x), \quad (\text{A}\cdot 22)$$

$$\begin{aligned} [\rho_a(q), \psi_{1-}(x)] &= [\rho_a(q), \psi_{2+}(x)] = [\rho_b(q), \psi_{1+}(x)] \\ &= [\rho_b(q), \psi_{2-}(x)] = 0. \end{aligned} \quad (\text{A}\cdot 23)$$

Noting eqs. (A·19)-(A·23), we express the field operators as³²

$$\psi_{1s}(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{ik_F x + i\theta_{1s}(x)}, \quad (\text{A}\cdot 24)$$

$$\psi_{2s}(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{-ik_F x + i\theta_{2s}(x)}, \quad (\text{A}\cdot 25)$$

where

$$\begin{aligned} \theta_{1+}(x) &= \theta_{1+}^Z + i\frac{\pi}{L} \sum_{q>0} \frac{e^{-\alpha q/2}}{q} \left(e^{-iq(x+\xi/2)} \rho_a(q) \right. \\ &\quad \left. - e^{iq(x+\xi/2)} \rho_a(-q) \right), \end{aligned} \quad (\text{A}\cdot 26)$$

$$\begin{aligned} \theta_{1-}(x) &= \theta_{1-}^Z + i\frac{\pi}{L} \sum_{q>0} \frac{e^{-\alpha q/2}}{q} \left(e^{-iq(x+\xi/2)} \rho_b(q) \right. \\ &\quad \left. - e^{iq(x+\xi/2)} \rho_b(-q) \right), \end{aligned} \quad (\text{A}\cdot 27)$$

$$\begin{aligned} \theta_{2+}(x) &= \theta_{2+}^Z + i\frac{\pi}{L} \sum_{q>0} \frac{e^{-\alpha q/2}}{q} \left(-e^{iq(x+\xi/2)} \rho_b(q) \right. \\ &\quad \left. + e^{-iq(x+\xi/2)} \rho_b(-q) \right), \end{aligned} \quad (\text{A}\cdot 28)$$

$$\begin{aligned} \theta_{2-}(x) &= \theta_{2-}^Z + i\frac{\pi}{L} \sum_{q>0} \frac{e^{-\alpha q/2}}{q} \left(-e^{iq(x+\xi/2)} \rho_a(q) \right. \\ &\quad \left. + e^{-iq(x+\xi/2)} \rho_a(-q) \right). \end{aligned} \quad (\text{A}\cdot 29)$$

Here, α is a positive infinitesimal and θ_{js}^Z ($j = 1, 2$) represents the zero-mode component to be determined. The phase field θ_{js} is related with the density operator ρ_{js} ,

$$\rho_{js}(x) = \frac{1}{2\pi} \frac{\partial \theta_{js}}{\partial x}. \quad (\text{A}\cdot 30)$$

For example, we find that

$$\begin{aligned} \rho_{1+}(x) &= \rho_{1+}^Z(x) + \frac{1}{2L} \sum_{q>0} \left(e^{-iq(x+\xi/2)} \rho_a(q) \right. \\ &\quad \left. + e^{iq(x+\xi/2)} \rho_a(-q) \right), \end{aligned} \quad (\text{A}\cdot 31)$$

$$\begin{aligned} \rho_{2-}(x) &= \rho_{2-}^Z(x) + \frac{1}{2L} \sum_{q>0} \left(e^{iq(x+\xi/2)} \rho_a(q) \right. \\ &\quad \left. + e^{-iq(x+\xi/2)} \rho_a(-q) \right), \end{aligned} \quad (\text{A}\cdot 32)$$

where

$$\rho_{js}^Z(x) = \frac{1}{2\pi} \frac{\partial \theta_{js}^Z}{\partial x}. \quad (\text{A}\cdot 33)$$

From eqs. (A·31) and (A·32), we naturally find that

$$\rho_{1+}^Z(x) = \rho_{2-}^Z(x) = \frac{\pi}{L} \left(N_a + \frac{\chi}{2\pi} \right). \quad (\text{A}\cdot 34)$$

We determin θ_{js}^Z so as to ensure the fermion commutation relation between $\psi_{j\pm}$ and $\psi_{j'\pm}$ and that between $\psi_{j\pm}$ and $\psi_{j'\pm}^\dagger$ ($j, j' = 1, 2$). The result is

$$\begin{aligned} \theta_{1+}^Z(x) &= \varphi_a + \frac{\pi}{L} \left(N_a + \frac{\chi}{2\pi} \right) \left(x + \frac{\xi}{2} \right) \\ &\quad + \frac{\pi}{2} (N_a - N_b), \end{aligned} \quad (\text{A}\cdot 35)$$

$$\begin{aligned} \theta_{2-}^Z(x) &= -\varphi_a + \frac{\pi}{L} \left(N_a + \frac{\chi}{2\pi} \right) \left(x + \frac{\xi}{2} \right) \\ &\quad + \frac{\pi}{2} (N_a - N_b), \end{aligned} \quad (\text{A}\cdot 36)$$

$$\begin{aligned} \theta_{1-}^Z(x) &= \varphi_b + \frac{\pi}{L} \left(N_b + \frac{\chi}{2\pi} \right) \left(x + \frac{\xi}{2} \right) \\ &\quad - \frac{\pi}{2} (N_a - N_b), \end{aligned} \quad (\text{A}\cdot 37)$$

$$\begin{aligned} \theta_{2+}^Z(x) &= -\varphi_b + \frac{\pi}{L} \left(N_b + \frac{\chi}{2\pi} \right) \left(x + \frac{\xi}{2} \right) \\ &\quad - \frac{\pi}{2} (N_a - N_b), \end{aligned} \quad (\text{A}\cdot 38)$$

where φ_a and φ_b are the zero-mode operators mentioned above. Strictly speaking, if we need to ensure the fermion commutation relation between $\psi_{j\pm}$ and $\psi_{j'\mp}$ and that between $\psi_{j\pm}$ and $\psi_{j'\mp}^\dagger$, Majorana fermions must be introduced in eqs. (A·24) and (A·25). However, since the Majorana fermions do not play an essential role in our argument, we neglect them in the following. For actual calculations, it is convenient to use the new phase fields:³¹

$$\Theta_+(x) = \frac{1}{2} \left(\theta_{1+}(x) + \theta_{1-}(x) - \theta_{2+}(x) - \theta_{2-}(x) \right), \quad (\text{A}\cdot 39)$$

$$\Theta_-(x) = \frac{1}{2} \left(\theta_{1+}(x) + \theta_{1-}(x) + \theta_{2+}(x) + \theta_{2-}(x) \right), \quad (\text{A}\cdot 40)$$

$$\Phi_+(x) = \frac{1}{2} \left(\theta_{1+}(x) - \theta_{1-}(x) - \theta_{2+}(x) + \theta_{2-}(x) \right), \quad (\text{A}\cdot 41)$$

$$\Phi_-(x) = \frac{1}{2} \left(\theta_{1+}(x) - \theta_{1-}(x) + \theta_{2+}(x) - \theta_{2-}(x) \right). \quad (\text{A}\cdot 42)$$

To express these phase fields compactly, we define φ_ρ , φ_σ , J and M as

$$\varphi_\rho = \varphi_a + \varphi_b, \quad (\text{A}\cdot 43)$$

$$\varphi_\sigma = \varphi_a - \varphi_b, \quad (\text{A}\cdot 44)$$

$$J = N_a + N_b, \quad (\text{A}\cdot 45)$$

$$M = N_a - N_b. \quad (\text{A}\cdot 46)$$

It is easy to show that $[J, \varphi_\rho] = [M, \varphi_\sigma] = 2i$ and $[J, \varphi_\sigma] = [M, \varphi_\rho] = 0$. Substituting eqs. (A·35)-(A·38) into the above expressions and using these operators, we

find that

$$\Theta_+(x) = \varphi_\rho + \theta_+(x), \quad (\text{A}\cdot 47)$$

$$\Theta_-(x) = \frac{\pi}{L} \left(J + \frac{\chi}{\pi} \right) \left(x + \frac{\xi}{2} \right) + \theta_-(x), \quad (\text{A}\cdot 48)$$

$$\Phi_+(x) = \frac{\pi}{L} M \left(x + L + \frac{\xi}{2} \right) + \phi_+(x), \quad (\text{A}\cdot 49)$$

$$\Phi_-(x) = \varphi_\sigma + \phi_-(x), \quad (\text{A}\cdot 50)$$

where

$$\begin{aligned} \theta_+(x) = & i \frac{\pi}{L} \sum_{q>0} \frac{e^{-\alpha q/2}}{q} \cos q \left(x + \frac{\xi}{2} \right) \\ & \times (\rho_a(q) - \rho_a(-q) + \rho_b(q) - \rho_b(-q)), \quad (\text{A}\cdot 51) \end{aligned}$$

$$\begin{aligned} \theta_-(x) = & \frac{\pi}{L} \sum_{q>0} \frac{e^{-\alpha q/2}}{q} \sin q \left(x + \frac{\xi}{2} \right) \\ & \times (\rho_a(q) + \rho_a(-q) + \rho_b(q) + \rho_b(-q)), \quad (\text{A}\cdot 52) \end{aligned}$$

$$\begin{aligned} \phi_+(x) = & \frac{\pi}{L} \sum_{q>0} \frac{e^{-\alpha q/2}}{q} \sin q \left(x + \frac{\xi}{2} \right) \\ & \times (\rho_a(q) + \rho_a(-q) - \rho_b(q) - \rho_b(-q)), \quad (\text{A}\cdot 53) \end{aligned}$$

$$\begin{aligned} \phi_-(x) = & i \frac{\pi}{L} \sum_{q>0} \frac{e^{-\alpha q/2}}{q} \cos q \left(x + \frac{\xi}{2} \right) \\ & \times (\rho_a(q) - \rho_a(-q) - \rho_b(q) + \rho_b(-q)). \quad (\text{A}\cdot 54) \end{aligned}$$

Using the phase fields, we rewrite eqs. (A·24) and (A·25) as^{31, 32}

$$\psi_{1s}(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{ik_F x + \frac{1}{2}(\Theta_+(x) + \Theta_-(x) + s\Phi_+(x) + s\Phi_-(x))}, \quad (\text{A}\cdot 55)$$

$$\psi_{2s}(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{-ik_F x + \frac{1}{2}(-\Theta_+(x) + \Theta_-(x) - s\Phi_+(x) + s\Phi_-(x))}. \quad (\text{A}\cdot 56)$$

We note that the density operators with $q \neq 0$ and H_0 satisfy following relations

$$[\rho_a(q), H_0] = -v_F q \rho_a(q), \quad (\text{A}\cdot 57)$$

$$[\rho_b(q), H_0] = -v_F q \rho_b(q). \quad (\text{A}\cdot 58)$$

We separate the nonzero-mode component H_0^{NZ} from H_0 . From eqs. (A·57) and (A·58) we find that

$$H_0^{\text{NZ}} = \frac{\pi v_F}{L} \sum_{q>0} (\rho_a(q) \rho_a(-q) + \rho_b(q) \rho_b(-q)). \quad (\text{A}\cdot 59)$$

Next we consider the zero-mode component H_0^Z . The energy cost due to the excess charge $N_a (> 0)$ is obtained as

$$\begin{aligned} \Delta E(N_a) = & \sum_{n=0}^{N_a-1} v_F \left(\frac{\pi(n+1/2)}{L+\xi} + \frac{\chi}{2(L+\xi)} \right) \\ \approx & \frac{v_F \pi}{2L} N_a^2 + \frac{v_F \chi}{2L} N_a. \quad (\text{A}\cdot 60) \end{aligned}$$

This expression also holds for negative N_a . Similarly, we obtain the energy cost due to N_b . Adding $\Delta E(N_a)$,

$\Delta E(N_b)$ and the χ -dependent part of the ground-state energy $(v_F/4\pi)(\chi^2/L)$, we obtain

$$H_0^Z = \frac{v_F \pi}{2L} \left(\left(N_a + \frac{\chi}{2\pi} \right)^2 + \left(N_b + \frac{\chi}{2\pi} \right)^2 \right). \quad (\text{A}\cdot 61)$$

We have expressed the field operators and the Hamiltonian in terms of the density operators. In order to simplify the expressions, we introduce operators for $q > 0$:

$$\alpha_q = \sqrt{\frac{\pi}{2qL}} (\rho_a(-q) + \rho_b(-q)), \quad (\text{A}\cdot 62)$$

$$\alpha_q^\dagger = \sqrt{\frac{\pi}{2qL}} (\rho_a(q) + \rho_b(q)), \quad (\text{A}\cdot 63)$$

$$\beta_q = \sqrt{\frac{\pi}{2qL}} (\rho_a(-q) - \rho_b(-q)), \quad (\text{A}\cdot 64)$$

$$\beta_q^\dagger = \sqrt{\frac{\pi}{2qL}} (\rho_a(q) - \rho_b(q)). \quad (\text{A}\cdot 65)$$

They satisfy the boson commutation relation. Using the boson operators, we rewrite the phase fields defined in eqs. (A·51)-(A·53) as

$$\theta_+(x) = i \sum_{q>0} \sqrt{\frac{2\pi}{qL}} e^{-\alpha q/2} \cos q \left(x + \frac{\xi}{2} \right) (\alpha_q^\dagger - \alpha_q), \quad (\text{A}\cdot 66)$$

$$\theta_-(x) = \sum_{q>0} \sqrt{\frac{2\pi}{qL}} e^{-\alpha q/2} \sin q \left(x + \frac{\xi}{2} \right) (\alpha_q^\dagger + \alpha_q), \quad (\text{A}\cdot 67)$$

$$\phi_+(x) = \sum_{q>0} \sqrt{\frac{2\pi}{qL}} e^{-\alpha q/2} \sin q \left(x + \frac{\xi}{2} \right) (\beta_q^\dagger + \beta_q), \quad (\text{A}\cdot 68)$$

$$\phi_-(x) = i \sum_{q>0} \sqrt{\frac{2\pi}{qL}} e^{-\alpha q/2} \cos q \left(x + \frac{\xi}{2} \right) (\beta_q^\dagger - \beta_q). \quad (\text{A}\cdot 69)$$

Similarly, the Hamiltonian is expressed as

$$H_0 = \frac{v_F \pi}{4L} \left(\left(J + \frac{\chi}{\pi} \right)^2 + M^2 \right) + \sum_{q>0} v_F q (\alpha_q^\dagger \alpha_q + \beta_q^\dagger \beta_q). \quad (\text{A}\cdot 70)$$

Note that since $J = N_a + N_b$ and $M = N_a - N_b$, then J must be even (odd) if M is even (odd). That is, $J + M = \text{even}$.

We take account of electron-electron interactions in the following. We employ a model interaction Hamiltonian H_{int} given as follows:

$$\begin{aligned} H_{\text{int}} = & 2\pi v_F \int_0^L dx \sum_{s,s'} (g_2 \delta_{s,s'} + g_2' \delta_{s,-s'}) \rho_{1s}(x) \rho_{2s'}(x) \\ & + \pi v_F \int_0^L dx \sum_{s,s'} (g_4 \delta_{s,s'} + g_4' \delta_{s,-s'}) \\ & \times (\rho_{1s}(x) \rho_{1s'}(x) + \rho_{2s}(x) \rho_{2s'}(x)). \quad (\text{A}\cdot 71) \end{aligned}$$

In terms of $\rho_a(q)$ and $\rho_b(q)$, the interaction Hamiltonian is rewritten as

$$\begin{aligned} H_{\text{int}} = & -\frac{\pi v_F g_2}{L} \sum_q \rho_a(q) \rho_b(q) \\ & + \frac{\pi v_F g_2'}{2L} \sum_q (\rho_a(q) \rho_a(q) + \rho_b(q) \rho_b(q)) \\ & + \frac{\pi v_F g_4}{2L} \sum_q (\rho_a(q) \rho_a(-q) + \rho_b(q) \rho_b(-q)) \\ & + \frac{\pi v_F g_4'}{L} \sum_q \rho_a(q) \rho_b(-q). \end{aligned} \quad (\text{A}\cdot 72)$$

We decompose H_{int} into the nonzero-mode component $H_{\text{int}}^{\text{NZ}}$ and the zero-mode component $H_{\text{int}}^{\text{Z}}$,

$$\begin{aligned} H_{\text{int}}^{\text{NZ}} = & -(g_2 + g_2') \sum_{q>0} \frac{v_F q}{2} (\alpha_q^\dagger \alpha_q^\dagger + \alpha_q \alpha_q) \\ & + (g_2 - g_2') \sum_{q>0} \frac{v_F q}{2} (\beta_q^\dagger \beta_q^\dagger + \beta_q \beta_q) \\ & + (g_4 + g_4') \sum_{q>0} v_F q \alpha_q^\dagger \alpha_q + (g_4 - g_4') \sum_{q>0} v_F q \beta_q^\dagger \beta_q, \end{aligned} \quad (\text{A}\cdot 73)$$

$$\begin{aligned} H_{\text{int}}^{\text{Z}} = & -\frac{\pi v_F}{4L} (g_2 + g_2' - g_4 - g_4') \left(J + \frac{\chi}{\pi} \right)^2 \\ & + \frac{\pi v_F}{4L} (g_2 - g_2' + g_4 - g_4') M^2. \end{aligned} \quad (\text{A}\cdot 74)$$

The nonzero-mode part $H^{\text{NZ}} \equiv H_0^{\text{NZ}} + H_{\text{int}}^{\text{NZ}}$ is diagonalized as

$$H^{\text{NZ}} = \sum_{q>0} (v_\rho q a_q^\dagger a_q + v_\sigma q b_q^\dagger b_q), \quad (\text{A}\cdot 75)$$

where

$$v_{\rho(\sigma)} = v_F \sqrt{(1 + g_4 \pm g_4')^2 - (g_2 \pm g_2')^2}. \quad (\text{A}\cdot 76)$$

Here, the new boson operators are given by

$$a_q = \alpha_q \cosh \lambda_\rho - \alpha_q^\dagger \sinh \lambda_\rho, \quad (\text{A}\cdot 77)$$

$$b_q = \beta_q \cosh \lambda_\sigma + \beta_q^\dagger \sinh \lambda_\sigma, \quad (\text{A}\cdot 78)$$

with

$$\tanh(2\lambda_{\rho(\sigma)}) = \frac{g_2 \pm g_2'}{1 + g_4 \pm g_4'}. \quad (\text{A}\cdot 79)$$

The zero-mode part $H^{\text{Z}} \equiv H_0^{\text{Z}} + H_{\text{int}}^{\text{Z}}$ is expressed as

$$H^{\text{Z}} = \frac{\pi}{4L} \left(v_\rho K_\rho \left(J + \frac{\chi}{\pi} \right)^2 + \frac{v_\sigma}{K_\sigma} M^2 \right), \quad (\text{A}\cdot 80)$$

where $K_{\rho(\sigma)}$ is the correlation exponent

$$K_{\rho(\sigma)} = \sqrt{\frac{1 + (g_4 \pm g_4') - (g_2 \pm g_2')}{1 + (g_4 \pm g_4') + (g_2 \pm g_2')}}. \quad (\text{A}\cdot 81)$$

Consequently, the total Hamiltonian is

$$\begin{aligned} H = & \frac{\pi}{4L} \left(v_\rho K_\rho \left(J + \frac{\chi}{\pi} \right)^2 + \frac{v_\sigma}{K_\sigma} M^2 \right) \\ & + \sum_{q>0} (v_\rho q a_q^\dagger a_q + v_\sigma q b_q^\dagger b_q). \end{aligned} \quad (\text{A}\cdot 82)$$

Finally, we rewrite θ_\pm and ϕ_\pm in terms of a_q, a_q^\dagger, b_q and b_q^\dagger as

$$\theta_+(x) = i\sqrt{K_\rho} \sum_{q>0} \sqrt{\frac{2\pi}{qL}} e^{-\alpha q/2} \cos q \left(x + \frac{\xi}{2} \right) (a_q^\dagger - a_q), \quad (\text{A}\cdot 83)$$

$$\theta_-(x) = \frac{1}{\sqrt{K_\rho}} \sum_{q>0} \sqrt{\frac{2\pi}{qL}} e^{-\alpha q/2} \sin q \left(x + \frac{\xi}{2} \right) (a_q^\dagger + a_q), \quad (\text{A}\cdot 84)$$

$$\phi_+(x) = \sqrt{K_\sigma} \sum_{q>0} \sqrt{\frac{2\pi}{qL}} e^{-\alpha q/2} \sin q \left(x + \frac{\xi}{2} \right) (b_q^\dagger + b_q), \quad (\text{A}\cdot 85)$$

$$\phi_-(x) = i\frac{1}{K_\sigma} \sum_{q>0} \sqrt{\frac{2\pi}{qL}} e^{-\alpha q/2} \cos q \left(x + \frac{\xi}{2} \right) (b_q^\dagger - b_q). \quad (\text{A}\cdot 86)$$

Appendix B: Derivation of $S_{\text{eff}}^{\text{NZ}}$

Let us consider Z^{NZ} defined as

$$\begin{aligned} Z^{\text{NZ}} = & \int \prod_{q>0} \mathcal{D}a_q^\dagger \mathcal{D}a_q \mathcal{D}b_q^\dagger \mathcal{D}b_q \\ & \times \exp \left(-S^{\text{NZ}}[\{a_q^\dagger, a_q, b_q^\dagger, b_q\}] - S^{\text{B}}[\{a_q^\dagger, a_q, b_q^\dagger, b_q\}] \right). \end{aligned} \quad (\text{B}\cdot 1)$$

We can rewrite Z^{NZ} as

$$\begin{aligned} Z^{\text{NZ}} = & \int \prod_{q>0} \mathcal{D}a_q^\dagger \mathcal{D}a_q \mathcal{D}b_q^\dagger \mathcal{D}b_q \int \mathcal{D}\bar{\theta} \mathcal{D}\tilde{\theta} \mathcal{D}\bar{\phi} \mathcal{D}\tilde{\phi} \\ & \times \delta \left(\bar{\theta} - \frac{1}{2} (\theta_+(L) + \theta_+(0)) \right) \\ & \times \delta \left(\tilde{\theta} - (\theta_+(L) - \theta_+(0)) \right) \\ & \times \delta \left(\bar{\phi} - \frac{1}{2} (\phi_+(L) + \phi_+(0)) \right) \\ & \times \delta \left(\tilde{\phi} - (\phi_+(L) - \phi_+(0)) \right) \\ & \times \exp \left(-S^{\text{NZ}}[\{a_q^\dagger, a_q, b_q^\dagger, b_q\}] - S^{\text{B}}[\{\bar{\theta}, \tilde{\theta}, \bar{\phi}, \tilde{\phi}\}] \right) \\ & \times \int \mathcal{D}\bar{\theta} \mathcal{D}\tilde{\theta} \mathcal{D}\bar{\phi} \mathcal{D}\tilde{\phi} \exp \left(-S^{\text{B}}[\{\bar{\theta}, \tilde{\theta}, \bar{\phi}, \tilde{\phi}\}] \right) \\ & \times \int \mathcal{D}\bar{\lambda}_\rho \mathcal{D}\tilde{\lambda}_\rho \mathcal{D}\bar{\lambda}_\sigma \mathcal{D}\tilde{\lambda}_\sigma \int \prod_{q>0} \mathcal{D}a_q^\dagger \mathcal{D}a_q \mathcal{D}b_q^\dagger \mathcal{D}b_q \\ & \times \exp \left(-S^{\text{NZ}}[\{a_q^\dagger, a_q, b_q^\dagger, b_q\}] \right. \\ & \left. + i\mathcal{P}[\{a_q^\dagger, a_q, b_q^\dagger, b_q\}, \bar{\theta}, \tilde{\theta}, \bar{\phi}, \tilde{\phi}, \bar{\lambda}_\rho, \tilde{\lambda}_\rho, \bar{\lambda}_\sigma, \tilde{\lambda}_\sigma] \right). \end{aligned} \quad (\text{B}\cdot 2)$$

where

$$\begin{aligned} \mathcal{P} = & \int d\tau \left[\bar{\lambda}_\rho(\tau) \left(\bar{\theta}(\tau) - \frac{1}{2} (\theta_+(L, \tau) + \theta_+(0, \tau)) \right) \right. \\ & \left. + \tilde{\lambda}_\rho(\tau) \left(\tilde{\theta}(\tau) - (\theta_+(L, \tau) - \theta_+(0, \tau)) \right) \right] \end{aligned}$$

$$+ \bar{\lambda}_\sigma(\tau) \left(\bar{\phi}(\tau) - \frac{1}{2} (\phi_+(L, \tau) + \phi_+(0, \tau)) \right) \\ + \tilde{\lambda}_\sigma(\tau) \left(\tilde{\phi}(\tau) - (\phi_+(L, \tau) - \phi_+(0, \tau)) \right) \Big]. \quad (\text{B}\cdot 3)$$

Note that $\theta_+(L)$ and $\theta_+(0)$ in eqs. (B·2) and (B·3) are functions of $\{a_q^\dagger, a_q\}$ as shown in eq. (14). Similarly, $\phi_+(L)$ and $\phi_+(0)$ are functions of $\{b_q^\dagger, b_q\}$. Integration over a_q, a_q^\dagger, b_q and b_q^\dagger yields

$$\int \mathcal{D}\bar{\lambda}_\rho \mathcal{D}\tilde{\lambda}_\rho \mathcal{D}\bar{\lambda}_\sigma \mathcal{D}\tilde{\lambda}_\sigma \int \prod_{q>0} \mathcal{D}a_q^\dagger \mathcal{D}a_q \mathcal{D}b_q^\dagger \mathcal{D}b_q \\ \times \exp \left(-S^{\text{NZ}}[\{a_q^\dagger, a_q, b_q^\dagger, b_q\}] \right. \\ \left. \times i\mathcal{P}[\{a_q^\dagger, a_q, b_q^\dagger, b_q\}, \bar{\theta}, \tilde{\theta}, \bar{\phi}, \tilde{\phi}, \bar{\lambda}_\rho, \tilde{\lambda}_\rho, \bar{\lambda}_\sigma, \tilde{\lambda}_\sigma] \right) \\ = \int \mathcal{D}\bar{\lambda}_\rho \mathcal{D}\tilde{\lambda}_\rho \mathcal{D}\bar{\lambda}_\sigma \mathcal{D}\tilde{\lambda}_\sigma \\ \times \exp \left(-\frac{\pi K_\rho}{2\beta} \sum_\omega \bar{J}_\rho(\omega) \bar{\lambda}_\rho(\omega) \bar{\lambda}_\rho(-\omega) \right. \\ \left. - i\frac{1}{\beta} \sum_\omega \bar{\theta}(\omega) \bar{\lambda}_\rho(-\omega) \right. \\ \left. - \frac{2\pi K_\rho}{\beta} \sum_\omega \tilde{J}_\rho(\omega) \tilde{\lambda}_\rho(\omega) \tilde{\lambda}_\rho(-\omega) \right. \\ \left. - i\frac{1}{\beta} \sum_\omega \tilde{\theta}(\omega) \tilde{\lambda}_\rho(-\omega) \right. \\ \left. - \frac{\pi K_\sigma}{2\beta} \sum_\omega \bar{J}_\sigma(\omega) \bar{\lambda}_\sigma(\omega) \bar{\lambda}_\sigma(-\omega) \right. \\ \left. - i\frac{1}{\beta} \sum_\omega \bar{\phi}(\omega) \bar{\lambda}_\sigma(-\omega) \right. \\ \left. - \frac{2\pi K_\sigma}{\beta} \sum_\omega \tilde{J}_\sigma(\omega) \tilde{\lambda}_\sigma(\omega) \tilde{\lambda}_\sigma(-\omega) \right. \\ \left. - i\frac{1}{\beta} \sum_\omega \tilde{\phi}(\omega) \tilde{\lambda}_\sigma(-\omega) \right), \quad (\text{B}\cdot 4)$$

where

$$\bar{J}_\rho(\omega) = \frac{1}{2\omega \sinh\left(\frac{L\omega}{v_\rho}\right)} \left\{ \cosh\left(\frac{L\omega}{v_\rho}\right) \right. \\ \left. + \cosh\left(\frac{(L-\xi)\omega}{v_\rho}\right) + \cosh\left(\frac{\xi\omega}{v_\rho}\right) + 1 \right\} - \frac{2v_\rho}{L\omega^2}, \quad (\text{B}\cdot 5)$$

$$\tilde{J}_\rho(\omega) = \frac{1}{2\omega \sinh\left(\frac{L\omega}{v_\rho}\right)} \left\{ \cosh\left(\frac{L\omega}{v_\rho}\right) \right. \\ \left. + \cosh\left(\frac{(L-\xi)\omega}{v_\rho}\right) - \cosh\left(\frac{\xi\omega}{v_\rho}\right) - 1 \right\}, \quad (\text{B}\cdot 6)$$

$$\bar{J}_\sigma(\omega) = \frac{1}{2\omega \sinh\left(\frac{L\omega}{v_\sigma}\right)} \left\{ \cosh\left(\frac{L\omega}{v_\sigma}\right) \right.$$

$$\left. - \cosh\left(\frac{(L-\xi)\omega}{v_\sigma}\right) - \cosh\left(\frac{\xi\omega}{v_\sigma}\right) + 1 \right\}, \quad (\text{B}\cdot 7)$$

$$\tilde{J}_\sigma(\omega) = \frac{1}{2\omega \sinh\left(\frac{L\omega}{v_\sigma}\right)} \left\{ \cosh\left(\frac{L\omega}{v_\sigma}\right) \right. \\ \left. - \cosh\left(\frac{(L-\xi)\omega}{v_\sigma}\right) + \cosh\left(\frac{\xi\omega}{v_\sigma}\right) - 1 \right\}. \quad (\text{B}\cdot 8)$$

Integrating out $\bar{\lambda}_\rho, \tilde{\lambda}_\rho, \bar{\lambda}_\sigma$ and $\tilde{\lambda}_\sigma$, we find that

$$\int \mathcal{D}\bar{\lambda}_\rho \mathcal{D}\tilde{\lambda}_\rho \mathcal{D}\bar{\lambda}_\sigma \mathcal{D}\tilde{\lambda}_\sigma \int \prod_{q>0} \mathcal{D}a_q^\dagger \mathcal{D}a_q \mathcal{D}b_q^\dagger \mathcal{D}b_q \\ \times \exp(-S^{\text{NZ}} + i\mathcal{P}) \\ \propto \exp(-S_{\text{eff}}^{\text{NZ}}), \quad (\text{B}\cdot 9)$$

where

$$S_{\text{eff}}^{\text{NZ}} = \frac{1}{2\pi K_\rho \beta} \sum_\omega \bar{J}_\rho^{-1}(\omega) \bar{\theta}(\omega) \bar{\theta}(-\omega) \\ + \frac{1}{8\pi K_\rho \beta} \sum_\omega \tilde{J}_\rho^{-1}(\omega) \tilde{\theta}(\omega) \tilde{\theta}(-\omega) \\ + \frac{1}{2\pi K_\sigma \beta} \sum_\omega \bar{J}_\sigma^{-1}(\omega) \bar{\phi}(\omega) \bar{\phi}(-\omega) \\ + \frac{1}{8\pi K_\sigma \beta} \sum_\omega \tilde{J}_\sigma^{-1}(\omega) \tilde{\phi}(\omega) \tilde{\phi}(-\omega). \quad (\text{B}\cdot 10)$$

We thus find that

$$Z^{\text{NZ}} \propto \int \mathcal{D}\bar{\theta} \mathcal{D}\tilde{\theta} \mathcal{D}\bar{\phi} \mathcal{D}\tilde{\phi} \\ \times \exp \left(-S_{\text{eff}}^{\text{NZ}}[\bar{\theta}, \tilde{\theta}, \bar{\phi}, \tilde{\phi}] - S^{\text{B}}[\bar{\theta}, \tilde{\theta}, \bar{\phi}, \tilde{\phi}] \right). \quad (\text{B}\cdot 11)$$

It is clear that the effective action given in eq. (B·10) satisfies eq. (36).

Appendix C: Derivation of S^Γ

We assume that our system is symmetric with respect to the left contact at $x = 0$ and the right contact at $x = L$. For simplicity, we treat only the coupling at the left contact between the TL liquid and the left superconductor, so that we do not explicitly express the subscript L, indicating the left superconductor, in the following derivation.

We first rewrite the tunneling Hamiltonian given in eq. (93) using the Bogoliubov transformation,

$$H^{\text{T}} = \frac{1}{\sqrt{V}} \sum_{k,s} \\ \times \left[d_{k,+}^\dagger \left(u_k \int dx t_k(x) \psi_+(x) \right. \right. \\ \left. \left. + v_k e^{i\chi_1} \int dx t_{-k}^*(x) \psi_-^\dagger(x) \right) \right. \\ \left. + d_{-k,-}^\dagger \left(u_k \int dx t_{-k}(x) \psi_-(x) \right) \right]$$

$$\begin{aligned}
& -v_k e^{i\chi_1} \int dx t_k^*(x) \psi_+^\dagger(x) \\
& + \left(v_k e^{-i\chi_1} \int dx t_{-k}(x) \psi_-(x) \right. \\
& \quad \left. + u_k \int dx t_k^*(x) \psi_+^\dagger(x) \right) d_{k,+} \\
& + \left(-v_k e^{-i\chi_1} \int dx t_k(x) \psi_+(x) \right. \\
& \quad \left. + u_k \int dx t_{-k}^*(x) \psi_-^\dagger(x) \right) d_{-k,-} \Big]. \quad (\text{C.1})
\end{aligned}$$

The action S^Γ is obtained by integrating out the electron field in the left superconductor,

$$\exp(-S^\Gamma) \propto \prod_{k,s} \mathcal{D}d_{k,s}^\dagger \mathcal{D}d_{k,s} \exp(-S^S - S^T), \quad (\text{C.2})$$

where

$$\begin{aligned}
S^S = \frac{1}{\beta} \sum_{\epsilon} (-i\epsilon + E_k) & \left(d_{k,+}^\dagger(\epsilon) d_{k,+}(\epsilon) \right. \\
& \left. + d_{-k,-}^\dagger(\epsilon) d_{-k,-}(\epsilon) \right), \quad (\text{C.3})
\end{aligned}$$

$$S^T = \int_0^\beta d\tau H^T(\tau). \quad (\text{C.4})$$

Here, ϵ denotes the fermion Matsubara frequency. After the integration, we find that

$$\begin{aligned}
S^\Gamma = \frac{1}{\beta} \sum_{\epsilon} \frac{1}{V} \sum_k \frac{u_k v_k}{-i\epsilon + E_k} \\
\times \left(e^{-i\chi_1} \int dx t_{-k}(x) \psi_-(x, -\epsilon) \int dy t_k(y) \psi_+(y, \epsilon) \right. \\
+ e^{i\chi_1} \int dx t_k^*(x) \psi_+^\dagger(x, \epsilon) \int dy t_{-k}^*(y) \psi_-^\dagger(y, -\epsilon) \\
- e^{-i\chi_1} \int dx t_k(x) \psi_+(x, -\epsilon) \int dy t_{-k}(y) \psi_-(y, \epsilon) \\
\left. - e^{i\chi_1} \int dx t_{-k}^*(x) \psi_-^\dagger(x, \epsilon) \int dy t_k^*(y) \psi_+^\dagger(y, -\epsilon) \right). \quad (\text{C.5})
\end{aligned}$$

In deriving eq. (C.5), we neglected irrelevant terms which do not depend on the macroscopic phase χ_1 . Equation (C.5) is simplified to

$$\begin{aligned}
S^\Gamma = \frac{1}{\beta} \sum_{\epsilon} \frac{1}{V} \sum_k \frac{2E_k}{\epsilon^2 + E_k^2} u_k v_k \\
\times \left(e^{-i\chi_1} \int dx \int dy t_{-k}(x) t_k(y) \psi_-(x, -\epsilon) \psi_+(y, \epsilon) \right. \\
\left. + \text{h.c.} \right). \quad (\text{C.6})
\end{aligned}$$

We assume that the tunneling-matrix element satisfies

$$\langle t_k(x) t_{-k}(y) \rangle_{k_F} = \tilde{\Gamma}(x) \delta(x - y), \quad (\text{C.7})$$

where $\langle \cdots \rangle_{k_F}$ denotes the average over k on the Fermi surface. Note that $\tilde{\Gamma}(x)$ has nonzero values only in the vicinity of $x = 0$ and vanishes when $\alpha_T < x$. If we carry

out the integrations over x and y in eq. (C.6) after averaging over k on the Fermi surface, the cross terms between $\psi_{1\pm}(x)$ and $\psi_{2\mp}(x)$ vanish due to the presence of a spatially oscillating factor of $e^{\pm i 2k_F x}$. After the summation over k , we find that

$$\begin{aligned}
S^\Gamma \approx \frac{1}{\beta} \sum_{\epsilon} \frac{\Gamma \Delta}{\sqrt{\Delta^2 + \epsilon^2}} & \left(e^{-i\chi_1} \left(\psi_{1-}(0, -\epsilon) \psi_{2+}(0, \epsilon) \right. \right. \\
& \left. \left. + \psi_{2-}(0, -\epsilon) \psi_{1+}(0, \epsilon) \right) + \text{h.c.} \right), \quad (\text{C.8})
\end{aligned}$$

where $\Gamma = \pi N(0) \int dx \tilde{\Gamma}(x)$ ($N(0)$: density of states at the Fermi level). In the imaginary-time representation, S^Γ is rewritten as

$$\begin{aligned}
S^\Gamma = \int_0^\beta d\tau_1 d\tau_2 Q(\tau_1 - \tau_2) & \left(e^{-i\chi_1} \left(\psi_{1-}(0, \tau_1) \psi_{2+}(0, \tau_2) \right. \right. \\
& \left. \left. + \psi_{2-}(0, \tau_1) \psi_{1+}(0, \tau_2) \right) + \text{h.c.} \right), \quad (\text{C.9})
\end{aligned}$$

where

$$Q(\tau) = \frac{1}{\beta} \sum_{\epsilon} \frac{\Gamma \Delta}{\sqrt{\Delta^2 + \epsilon^2}} e^{-i\epsilon \tau}. \quad (\text{C.10})$$

Note that we are interested in the case of $T \ll \Delta$, where $Q(\tau)$ is approximated as

$$Q(\tau) = \frac{1}{\pi} \int_{\Delta}^{\infty} dz \frac{\Gamma \Delta}{\sqrt{z^2 - \Delta^2}} e^{-|\tau|z}. \quad (\text{C.11})$$

This indicates that $Q(\tau)$ decays exponentially with a characteristic time scale of the order of Δ^{-1} .

Adding the term describing the coupling at the right contact, we finally obtain

$$\begin{aligned}
S^\Gamma = \int_0^\beta d\tau_1 d\tau_2 Q(\tau_1 - \tau_2) \\
\times \left(e^{-i\chi_1} \left(\psi_{1-}(0, \tau_1) \psi_{2+}(0, \tau_2) \right. \right. \\
+ \psi_{2-}(0, \tau_1) \psi_{1+}(0, \tau_2) \Big) + \text{h.c.} \\
+ e^{-i\chi_2} \left(\psi_{1-}(L, \tau_1) \psi_{2+}(L, \tau_2) \right. \\
+ \psi_{2-}(L, \tau_1) \psi_{1+}(L, \tau_2) \Big) + \text{h.c.} \Big). \quad (\text{C.12})
\end{aligned}$$

- 1) H. Takayanagi and T. Kawakami: Phys. Rev. Lett. **54** (1985) 2449.
- 2) B. J. van Wees and H. Takayanagi: *Mesoscopic Electron Transport*, ed. L. L. Sohn, L. P. Kouwenhoven and G. Schön (Kluwer, Dordrecht, 1997) p. 469.
- 3) C. J. Lambert and R. Raimondi: J. Phys.: Condens. Matter **10** (1998) 901.
- 4) S. Tarucha, T. Honda and T. Saku: Solid State Commun. **94** (1995) 413.
- 5) A. Yacoby, H. L. Stormer, N. S. Wingreen, L. N. Pfeiffer, K. W. Baldwin and K. W. West: Phys. Rev. Lett. **77** (1996) 4612.
- 6) C. L. Kane and M. P. A. Fisher: Phys. Rev. Lett. **68** (1992) 1220.
- 7) C. L. Kane and M. P. A. Fisher: Phys. Rev. B **46** (1992) 7268.
- 8) C. L. Kane and M. P. A. Fisher: Phys. Rev. B **46** (1992) 15233.
- 9) A. Furusaki and N. Nagaosa: Phys. Rev. B **47** (1993) 3827.
- 10) A. Furusaki and N. Nagaosa: Phys. Rev. B **47** (1993) 4631.

- 11) K. A. Matveev, D. Yue and L. I. Glazman: Phys. Rev. Lett. **71** (1993) 3351.
- 12) N. Nagaosa and A. Furusaki: J. Phys. Soc. Jpn. **63** (1994) 413.
- 13) M. Fabrizio, A. O. Gogolin and S. Scheidl: Phys. Rev. Lett. **72** (1994) 2235.
- 14) A. Kawabata: J. Phys. Soc. Jpn. **63** (1994) 2047.
- 15) S. Tomonaga: Prog. Theor. Phys. **5** (1950) 544.
- 16) J. M. Luttinger: J. Math. Phys. **4** (1963) 1154.
- 17) F. D. M. Haldane: J. Phys. C **14** (1981) 2585.
- 18) M. P. A. Fisher: Phys. Rev. B **49** (1994) 14550.
- 19) R. Fazio, F. W. J. Hekking and A. A. Odintsov: Phys. Rev. Lett. **74** (1995) 1843.
- 20) R. Fazio, F. W. J. Hekking and A. A. Odintsov: Phys. Rev. B **53** (1996) 6653.
- 21) D. L. Maslov, M. Stone, P. M. Goldbart and D. Loss: Phys. Rev. B **53** (1996) 1548.
- 22) Y. Takane and Y. Koyama: J. Phys. Soc. Jpn. **65** (1996) 3630.
- 23) Y. Takane and Y. Koyama: J. Phys. Soc. Jpn. **66** (1997) 419.
- 24) Y. Takane: J. Phys. Soc. Jpn. **66** (1997) 537.
- 25) C. Winkelholz, R. Fazio, F. W. J. Hekking and G. Schön: Phys. Rev. Lett. **77** (1996) 3200.
- 26) I. Affleck, J. -S. Caux and A. M. Zagoskin: Phys. Rev. B **62** (2000) 1433.
- 27) T. Hirai, K. Kusakabe and Y. Tanaka: J. Phys. Chem. Solids **62** (2001) 257.
- 28) C. Ishii: Prog. Theor. Phys. **44** (1970) 1525.
- 29) M. Fabrizio and A. O. Gogolin: Phys. Rev. B **51** (1995) 17827.
- 30) P. G. de Gennes: *Superconductivity of Metals and Alloys* (Benjamin, New York, 1966), Chap. 5.
- 31) Y. Suzumura: Prog. Theor. Phys. **61** (1979) 1.
- 32) A. Luther and I. Peschel: Phys. Rev. B **9** (1974) 2911.
- 33) D. Loss: Phys. Rev. Lett. **69** (1992) 343.